Hue University Journal of Science: Natural Science
Vol. 128, No. 1D, 5–15, 2019

pISSN 1859-1388
eISSN 2615-9678

OPTIMALITY CONDITIONS FOR NON-LIPSCHITZ VECTOR PROBLEMS WITH INCLUSION CONSTRAINTS

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(*Received: 03 June 2019; Accepted: 04 November 2019*)

Abstract. We use the concept of approximation introduced by D.T. Luc et al. [1] as a generalized derivative for non-Lipschitz vector functions to consider vector problems with non-Lipschitz data under inclusion constraints. Some calculus of approximations are presented. A necessary optimality condition, a type of KKT condition, for local efficient solutions of the problems is established under an assumption on regularity. Applications and numerical examples are also given.

Keywords: non-Lipschitz vector problem, inclusion constraint, approximation, regularity, optimality condition

1 Introduction

Several problems in optimization, variational analysis and other fields of mathematics concern generalized equations of the form

$$0 \in F(x), \tag{1}$$

where $F: X \to Y$ is a set-valued map and X, Y are normed spaces. For instance, an inclusion constraint of the form

$$g(x) \in K, \tag{2}$$

where $g: X \to Y$ and $K \subset Y$, can be rewritten as (1) if we set

$$F(x) := g(x) - K.$$

A more typical example is a constraint system of equalities/inequalities

$$\begin{cases} g_i(x) \leq 0, i = 1, \dots, n. \\ h_j(x) = 0, j = 1, \dots, k, \end{cases} \tag{3}$$

where $g_i, h_j: X \to \mathbb{R}$. We can rewrite (3) as (1) by setting

$$g := (g_1, \dots, g_n, h_1, \cdots, h_k)$$
$$K := \mathbb{R}_+^n \times \{0_{\mathbb{R}^k}\}$$
$$F(x) := g(x) + K.$$

Vector optimization problems with inclusion constraint (1) have been studied by several authors [2–5]. In [2], objective functions are assumed locally Lipschitz. Second-order optimality conditions are considered in [3–5].

In this paper, we consider the vector problem

$$\min f(x) \text{ s.t. } 0 \in F(x), \tag{P}$$

where $f: X \to \mathbb{R}^m$ is a non-Lipschitz vector function. We shall use the concept of approximation introduced in [1] as generalized derivatives to investigate the problem. In the next section, we recall some properties of locally Lipschitz set-valued maps. The definition and some calculus of approximation are presented in Section 3. Section 4 is devoted to establishing a necessary optimality condition, a type of KKT's condition, for local efficient solutions to (P). Applications and examples are also given.

Let X be a normed space and let $A \subset X$. We denote the closed unit ball in X, the unit sphere in X, the closure of A, and the convex hull of A by B_X, S_X, clA, and coA, respectively.

2 Preliminaries

In this section, we assume that X, Y are Hilbert spaces and $F: X \to Y$ is a locally Lipschitz set-valued map with nonempty, closed and convex values. We recall that F is said to be locally Lipschitz at $\bar{x} \in X$ if there exist a neighborhood U of $\bar{x}$ and a positive number α satisfying

$$F(x_1) \subset F(x_2) + \alpha B_Y(0, \| x_1 - x_2 \|), \forall x_1, x_2 \in U.$$

The distant function of F is defined by

$$d_F(x) := \inf\{\| y \| : y \in F(x)\}, \forall x \in X.$$

It is a continuous function since F is locally Lipschitz. Let $x \in X$ be arbitrary. Set

$$Y_F^*(x) := \left\{ y^* \in Y^* : | \sup_{y \in F(x)} \langle y^*, y \rangle < +\infty \right\}$$

$$Y_F^* := \left\{ y^* \in Y^* : | \sup_{y \in F(x)} \langle y^*, y \rangle < +\infty, \forall x \in X \right\},$$

where Y^* is the topological dual space of Y.

Lemma 2.1 *$Y_F^*(x)$ is not dependent on x.*

Proof. Let $x \in X$ be arbitrary. Set

$$S := \{x' \in X : Y_F^*(x') = Y_F^*(x)\}.$$

We note that in a Hilbert space, the image of any ball under a continuous linear functional is bounded. Then, S is open since F is locally Lipschitz. Also by the locally Lipschitz assumption of F, every cluster point of S is contained in S. Hence, S is closed. Obviously, $S \neq \emptyset$. Hence, $S = X$ since every normed space is connected.

So, we have $Y_F^* = Y_F^*(x), \forall x \in X$. Note that Y_F^* is a convex cone. For $y^* \in Y_F^*$, define a support function of F by the rule

$$C_F(y^*, x) := \sup_{y \in F(x)} \langle y^*, y \rangle, \forall x \in X.$$

Since F is locally Lipschitz, it can be verified that $C_F(y^*, .)$ is locally Lipschitz, too.

We say that F has the Cl-property [2] if the set-valued map $(y^*, x) \in Y_F^* \times X \to \partial C_F(y^*, x) \subset X^*$ is u.s.c., where X^*, Y^* are endowed with the weak*-topology and X with the strong topology that is, if $x_n \to x$ in $X, y_n^* \xrightarrow{w^*} y^*$ in Y_F^*, $x_n^* \xrightarrow{w^*} x^*$ with $x_n^* \in \partial C_F(y_n^*, x_n)$, then $x^* \in \partial C_F(y^*, x)$. (Where $\partial C_F(y^*, x)$ denotes the Clarke generalized gradient of the support function $C_F(y^*, .)$ at x.)

We now recall and establish some useful properties of the distant function and support functions of F.

Lemma 2.2 [2, Proposition 3.1] *Assume that F has the Cl-property. If $d_F(x) > 0$, then there exists $y^* \in Y_F^* \cap S_Y^*$ such that*

$$\begin{cases} \partial d_F(x) \subset -\partial C_F(y^*, x). \\ d_F(x) = -C_F(y^*, x). \end{cases}$$

Lemma 2.3 *Let $\bar{x} \in X$ be arbitrary. If $\{y_n^*\} \subset Y_F^*, y_n^* \xrightarrow{w^*} y^*$, then*

$$C_F(y^*, \bar{x}) \leq \limsup_{n \to \infty} C_F(y_n^*, \bar{x}).$$

Proof. By the definition of support functions, one can find a sequence $\{y_m\} \subset F(\bar{x})$ with

$$\lim_{m \to \infty} \langle y^*, y_m \rangle = C_F(y^*, \bar{x}).$$

Since

$$\lim_{n \to \infty} \langle y_n^*, y_m \rangle = \langle y^*, y_m \rangle, \forall m,$$

one can choose a subsequence $\{y_{n_m}^*\}_m$ such that

$$\lim_{m \to \infty} \langle y_{n_m}^*, y_m \rangle = C_F(y^*, \bar{x})$$

which implies

$$C_F(y^*, \bar{x}) \leq \limsup_{m \to \infty} C_F(y_{n_m}^*, \bar{x}) \leq \limsup_{n \to \infty} C_F(y_n^*, \bar{x}).$$

Lemma 2.4 *For every $\bar{x} \in X, y^* \in Y_F^*$, there exists a neighborhood U of $\bar{x}$ satisfying*

$$C_F(y^*, \bar{x}) \leq C_F(y^*, x) + \alpha \| y^* \| \| x - \bar{x} \|, \forall x \in U,$$

where α is a Lipschitz constant of F at $\bar{x}$.

pISSN 1859-1388
eISSN 2615-9678

Proof. Since F is locally Lipschitz at $\bar{x}$, there exist a neighborhood U of $\bar{x}$ and a positive number α satisfying

$$F(\bar{x}) \subset F(x) + \alpha \parallel x - \bar{x} \parallel B_Y, \forall x \in U.$$

Let $y \in F(\bar{x})$ be arbitrary. One can find $y' \in F(x), u \in B_Y$ such that

$$y = y' + \alpha \parallel x - \bar{x} \parallel u.$$

Therefore,

$$\begin{aligned} \langle y^*, y \rangle &= \langle y^*, y' \rangle + \alpha \parallel x - \bar{x} \parallel \langle y^*, u \rangle \\ &\leq C_F(y^*, x) + \alpha \parallel x - \bar{x} \parallel \parallel y^* \parallel \end{aligned}$$

which implies

$$C_F(y^*, \bar{x}) \leq C_F(y^*, x) + \alpha \parallel y^* \parallel \parallel x - \bar{x} \parallel.$$

3 Approximation

Assume that X, Y are normed spaces. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of Y. We say that $\{A_n\}$ converges to $\{0\}$; denoted $A_n \to 0$, if

$$\forall \varepsilon > 0, \exists N: n \geq N \Rightarrow A_n \subset B_Y(0, \varepsilon).$$

Let $\bar{x}, u \in X$. A sequence $\{x_n\} \subset X$ is said to converge to $\bar{x}$ in the direction u, denoted $x_n \to_u \bar{x}$, if

$$\exists t_n \downarrow 0, u_n \to u \text{ such that } x_n = \bar{x} + t_n u_n, \forall n.$$

Let $r: X \to Y$. We say that r has limit 0 as x converges to 0 in direction u, denoted $\lim_{x \to_u 0} r(x) = 0$, if

$$\forall \{x_n\} \subset X, x_n \to_u 0 \Rightarrow r(x_n) \to \{0\}.$$

Denote the space of continuous linear mappings from X to Y by $L(X, Y)$. For $A \subset L(X, Y), y^* \in Y^*$ and $x \in X$, set $A(x) := \{a(x): a \in A\}, (y^* \circ A)(x) := y^*[A(x)]$.

Let $f: X \to Y$ and $\bar{x} \in X$. The following definition of approximation is a version of [1, Definition 3.1] with a minor change.

Definition 3.1 *A nonempty subset $A_f(\bar{x}) \subset L(X, Y)$ is called an approximation of f at $\bar{x} \in X$ if for every direction $u \neq 0$, there exists a set-valued map $r_u: X \to Y$ with $\lim_{x \to_u 0} r_u(x) = 0$, such that for every sequence $\{x_n\} \subset X$ converging to $\bar{x}$ in the direction u,*

$$f(x_n) \in f(\bar{x}) + A_f(\bar{x})(x_n - \bar{x}) + \parallel x_n - \bar{x} \parallel r_u(x_n - \bar{x}),$$

for n being sufficiently large.

The concept of approximation was first given by Jourani and Thibault [6] in a stronger form. It requires that

$$f(x) \in f(x') + A_f(\bar{x})(x - x') + \parallel x - x' \parallel r(x, x'),$$

where $r(x, x') \to 0$ as $x, x' \to \bar{x}$. Allali and Amahroq [7] give a weaker definition by taking $x' = \bar{x}$ in the above relation. It is clear from the above definitions that an approximation in the sense of Jourani and Thibault is an approximation in the sense of Allali and Amahroq, which, in its turn, is an approximation in the sense of Definition 3.1. However, the converse is not true in general as shown in [1]. The definition by Jourani and Thibault evokes the idea of strict derivatives and is very useful in the study of metric regularity and stability properties, while Definition 3.1 is more sensitive to the behavior of the function in directions and so it allows to treat certain questions such as existence conditions for a larger class of problems.

We note that the Clarke generalized gradient locally Lipschitz functions on Banach spaces [8] is an approximation in the sense of Allali and Amahroq. Hence, it is also an approximation in the sense of Definition 3.1. Now, we establish some basic calculus for approximations that will be needed in the sequel. The next two lemmas are immediate from Definition 3.1.

Lemma 3.1 *Let $f, g: X \to Y$. If f, g admit $A_f(\bar{x}), A_g(\bar{x})$, respectively, as approximations at $\bar{x} \in X$, then $f + g, (f, g)$ admit $A_f(\bar{x}) + A_g(\bar{x})$, $A_f(\bar{x}) \times A_g(\bar{x})$, respectively, as approximations at $\bar{x}$ (where $(f, g)(x) := \big(f(x), g(x)\big)$).*

Lemma 3.2 *Let $f: X \to Y$. If $A_f(\bar{x})$ is an approximation of f at $\bar{x} \in X$, then, for every $y^* \in Y^*$, $y^* \circ A_f(\bar{x})$ is an approximation of $y^* \circ f$ at $\bar{x}$.*

For a set-valued map $r: X \to Y$, we set

$M_r(x) := \sup\{\| z \| : z \in r(x)\}, \forall x \in X.$

Lemma 3.3 *Let $f: X \to Y$, $g: Y \to \mathbb{R}$. Assume that $A_f(\bar{x})$ is a bounded approximation of f at $\bar{x} \in X$ and g is differentiable at $\bar{y} := f(\bar{x})$. Then, $Dg(\bar{y}) \circ A_f(\bar{x})$ is an approximation of $g \circ f$ at $\bar{x}$.*

Proof. Since g is differentiable at $\bar{y}$, we have the following representation

$$g(y) = g(\bar{y}) + Dg(\bar{y})(y - \bar{y}) + \| y - \bar{y} \| s(y - \bar{y}),$$

where $s: Y \to \mathbb{R}$ satisfies $\lim_{z \to 0} s(z) = 0$. Let $u \in X\{0\}$ be arbitrary. By assumption, one can find a set-valued map $r_u: X \to Y$ with $\lim_{x \to_u 0} r(x) = 0$ such that for every sequence $x_n \to_u \bar{x}$, we have

$$f(x_n) \in f(\bar{x}) + A_f(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| r_u(x_n - \bar{x}),$$

for n being sufficiently large. Denote

$M := \sup\{\| \phi \| : \phi \in A_f(\bar{x})\}.$

We have

$$\begin{aligned} g[f(x_n)] &= g[f(\bar{x})] + Dg(\bar{y})[f(x_n) - f(\bar{x})] + \| f(x_n) - f(\bar{x}) \| s[f(x_n) - f(\bar{x})] \\ &\in g[f(\bar{x})] + Dg(\bar{y})[A_f(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| r_u(x_n - \bar{x})] + \\ &\quad + \| f(x_n) - f(\bar{x}) \| s[f(x_n) - f(\bar{x})] \\ &= g \circ f(\bar{x}) + [Dg(\bar{y}) \circ A_f(\bar{x})](x_n - \bar{x}) + \| x_n - \bar{x} \| [Dg(\bar{y}) \circ r_u](x_n - \bar{x}) + \\ &\quad + \| f(x_n) - f(\bar{x}) \| s[f(x_n) - f(\bar{x})] \\ &\subset g \circ f(\bar{x}) + [Dg(\bar{y}) \circ A_f](\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| [Dg(\bar{y}) \circ r_u](x_n - \bar{x}) + \\ &\quad + \| x_n - \bar{x} \| [0, M + M_{r_u}(x_n - \bar{x})] s[f(x_n) - f(\bar{x})] \\ &= g \circ f(\bar{x}) + [Dg(\bar{y}) \circ A_f(\bar{x})](x_n - \bar{x}) + \| x_n - \bar{x} \| r_u'(x_n - \bar{x}), \end{aligned}$$

where

$$r'_u(x) := Dg(\bar{y}) \circ r_u(x) + [0, M + M_{r_u}(x)] s'(x)$$

with $s'(x) := s[f(x + \bar{x}) - f(\bar{x})]$. It can be verified that $\lim_{x \to_u 0} r'_u(x) = 0$. The lemma is proved.

Let $f_1, f_2: X \to \mathbb{R}$. For every $x \in X$, put

$h(x) := \max\{f_1(x), f_2(x)\}$

and $J(x) := \{i | f_i(x) = h(x)\}$.

Lemma 3.4 *Assume that f_1, f_2 are continuous at $\bar{x} \in X$. If $A_{f_1}(\bar{x})$ and $A_{f_2}(\bar{x})$ are approximations of f_1 and f_2 at $\bar{x}$, respectively, then*

$$A_h(\bar{x}) := \bigcup_{i \in J(\bar{x})} A_{f_i}(\bar{x})$$

is an approximation of h at $\bar{x}$.

Proof. Let $u \in X\{0\}$ be arbitrary. By the definition of approximation, there exist maps $r_u, s_u: X \to \mathbb{R}$ with $\lim_{x \to_u 0} r_u(x) = 0$, $\lim_{x \to_u 0} s_u(x) = 0$ such that for every sequence $\{x_n\} \subset X$ converging to $\bar{x}$ in the direction u, one has

$$f_1(x_n) \in f_1(\bar{x}) + A_{f_1}(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| r_u(x_n - \bar{x}) \quad (4)$$

$$f_2(x_n) \in f_2(\bar{x}) + A_{f_2}(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| s_u(x_n - \bar{x}) \quad (5)$$

for n being sufficiently large. One of the following cases holds.

i) $J(\bar{x}) = \{1,2\}$. Set $t_u(x) = r_u(x) \cup s_u(x), \forall x \in X$. From (4) and (5), one has

$$h(x_n) \in h(\bar{x}) + \left(A_{f_1}(\bar{x}) \cup A_{f_2}(\bar{x})\right)(x_n - \bar{x}) + \| x_n - \bar{x} \| t_u(x_n - \bar{x})$$

for n being sufficiently large. Since $\lim_{x \to_u 0} t_u(x) = 0$, $A_{f_1}(\bar{x}) \cup A_{f_2}(\bar{x})$ is an approximate of h at $\bar{x}$.

ii) $J(\bar{x}) = \{1\}$. Since f_1, f_2 are continuous at $\bar{x}$, we have $f_1(x_n) > f_2(x_n)$ for n being sufficiently large. It implies that

$$h(x_n) \in h(\bar{x}) + A_{f_1}(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| r_u(x_n - \bar{x}).$$

Hence, $A_{f_1}(\bar{x})$ is an approximate of h at $\bar{x}$.

iii) $J(\bar{x}) = \{2\}$. Analogously, we have $A_{f_2}(\bar{x})$ is an approximate of h at $\bar{x}$. The lemma is proved.

Let $\phi: X \to \mathbb{R}$.

Lemma 3.5 *Assume that X is a reflexive space. If $\bar{x} \in X$ is a local minimum of ϕ and ϕ admits $A_\phi(\bar{x})$ as an approximation at $\bar{x}$, then*

$0 \in clco A_\phi(\bar{x})$.

pISSN 1859-1388
eISSN 2615-9678

Proof. Suppose, on the contrary, that $0 \notin clcoA_{\phi}(\bar{x})$. Since X is reflexive, by using the strong separation theorem, one can find a vector $u \in X\{0\}$ and a positive number ε satisfying

$$\langle x^*, u\rangle \leq -\varepsilon, \forall x^* \in A_{\phi}(\bar{x}).$$

Corresponding to the direction u, there exists a set-valued map $r_u: X \to \mathbb{R}$ with $\lim_{x \to_u 0} r_u(x) = 0$ such that for every sequence $x_n \to_u \bar{x}$, one has

$$f(x_n) \in f(\bar{x}) + A_{\phi}(\bar{x})(x_n - \bar{x}) + \| x_n - \bar{x} \| r_u(x_n - \bar{x})$$

for sufficiently large n. Since $x_n := \bar{x} + \frac{1}{n}u \to_u \bar{x}$, we get

$$n[f(x_n) - f(\bar{x})] \in A_{\phi}(\bar{x})(u) + \| u \| r_u(x_n - \bar{x}) \subset \left(-\infty, -\frac{\varepsilon}{2}\right]$$

for sufficiently large n. We have a contradiction.

4 Optimality condition

In this section, we assume that X, Y are Hilbert spaces and that $\mathbb{R}^m$ is ordered by a closed, convex cone C which is not a subspace. We denote the polar cone of C by C'; that is,

$$C' := \{z^* \in \mathbb{R}^m : \langle z^*, c\rangle \geq 0, \forall c \in C\}.$$

Let $f: X \to \mathbb{R}^m$ and let $F: X \to Y$ be locally Lipschitz with Y_F^* being weak* closed. We consider the problem

$$\min f(x) \text{ s.t. } 0 \in F(x). \qquad (P)$$

Set

$$S := \{x \in X : 0 \in F(x)\}.$$

We recall that a vector $\bar{x} \in S$ is called a local efficient solution of Problem (P) if there exists a neighborhood V of $\bar{x}$ such that

$$x \in S \cap V \Rightarrow f(x) \notin f(\bar{x}) - (C(C \cap -C)).$$

Problem (P) is said to be regular at a feasible point $\bar{x}$ [2] if there exist a neighborhood U of $\bar{x}$ and positive numbers δ, γ such that for every $x \in U, y^* \in Y_F^*, x^* \in \partial C_F(y^*, x)$, there exists $\eta \in B_X(0, \delta)$ satisfying

$$C_F(y^*, x) + \langle x^*, \eta\rangle \geq \gamma \| y^* \|. \qquad (6)$$

Firstly, we establish some results which will be used in the proof of the main result of the section. Let $A \subset \mathbb{R}^m$ be a nonempty set. Consider the support function of A

$$s(A, x) := \sup_{a \in A} \langle a, x\rangle.$$

For each $x \in \mathbb{R}^m$, we set

$$I(x) := \{a \in A : \langle a, x\rangle = s(A, x)\}.$$

Proposition 4.1 *Assume that A is compact. Then $s(A,.)$ is differentiable at $\bar{x} \in \mathbb{R}^m$ if and only if $I(\bar{x})$ is a singleton. In this case,*

$$\nabla s(A,.)(\bar{x}) = a$$

with a being the unique element of $I(\bar{x})$.

Proof. Since A is compact, $I(x) \neq \emptyset, \forall x$ and $s(A,.)$ is a convex function with the domain $\mathbb{R}^m$; consequently, $s(A,.)$ is locally Lipschitz on $\mathbb{R}^m$. Hence, by Rademacher's Theorem, $s(A,.)$ is differentiable almost everywhere (in the sense of Lebesgue measure) on $\mathbb{R}^m$. Denote the set of all points at which $s(A,.)$ is differentiable by M.

For the 'only if' part, assume that $s(A,.)$ is differentiable at $\bar{x}$. Let $\bar{a} \in I(\bar{x})$ and $v \in \mathbb{R}^m$ be arbitrary. We have

$$\begin{aligned}\langle \nabla s(A,.)(\bar{x}), v\rangle &= \lim_{t \downarrow 0} \frac{s(A, \bar{x} + tv) - s(A, \bar{x})}{t} \\ &= \lim_{t \downarrow 0} \frac{\sup_{a \in A}\langle a, \bar{x} + tv\rangle - \langle \bar{a}, \bar{x}\rangle}{t} \\ &\geq \lim_{t \downarrow 0} \frac{\langle \bar{a}, \bar{x} + tv\rangle - \langle \bar{a}, \bar{x}\rangle}{t} \\ &= \langle \bar{a}, v\rangle.\end{aligned}$$

This implies $\bar{a} = \nabla s(A,.)(\bar{x})$. Hence,

$$I(\bar{x}) = \{\nabla s(A,.)(\bar{x})\}.$$

For the 'if' part, assume that $I(\bar{x})$ is a singleton and its unique element is denoted by $\bar{a}$. Firstly, we show that the set-valued map $I: x \to$

$I(x)$ is u.s.c. at $\bar{x}$. Indeed, suppose the contrary, then one can find a number $\varepsilon > 0$ and a sequence $\{x_n\}$ converging to $\bar{x}$ such that

$I(x_n) \not\subset (\bar{a}, \varepsilon)$.

Let $a_n \in I(x_n)\backslash B(\bar{a}, \varepsilon)$. Since A is compact, we may assume that $a_n \to a$, for some $a \in A$ with $a \neq \bar{a}$. Since $\langle a_n, x_n\rangle \geq \langle \bar{a}, x_n\rangle$, taking the limit, we have $\langle a, \bar{x}\rangle \geq \langle \bar{a}, \bar{x}\rangle$. Hence, $\langle a, \bar{x}\rangle = s(A, \bar{x})$, which implies $a = \bar{a}$. We get a contradiction.

Now, let $\{x_n\} \subset M$ such that $x_n \to \bar{x}, \nabla s(A,.)(x_n) \to x^*$, for some $x^* \in \mathbb{R}^m$. From the proof of the 'only if' part, we have $I(x_n) = \{\nabla s(A,.)(x_n)\}$. Then, the upper semicontinuity of I at $\bar{x}$ implies $x^* = \bar{a}$, and consequently, $\partial s(A,.)(\bar{x}) = \{\bar{a}\} = I(\bar{x})$. Therefore, $s(A,.)$ is differentiable at $\bar{x}$ and $\nabla s(A,.)(\bar{x}) = \bar{a}$. The proof is complete.

Let $a \in \mathbb{R}^m$. We define a set-valued map $\Phi: X \to \mathbb{R}^m$ as follows

$\Phi(x) := f(x) + a + C$.

Lemma 4.1 *We have*

$$d_\Phi(x) = [s(C' \cap B_{\mathbb{R}^m},.) \circ (f + a)](x), \forall x \in X.$$

If $d_\Phi(x) > 0$, then there exists a unique element $z^* \in C' \cap B_{\mathbb{R}^m}$ such that

$$d_\Phi(x) = \langle z^*, f(x) + a\rangle.$$

Furthermore, $\| z^* \| = 1$.

Proof. Firstly, we prove that, for every $x \in X$,

$$d_\Phi(x) = \max_{y^* \in -C' \cap B_{\mathbb{R}^m}} - \sup_{y \in \Phi(x)} \langle y^*, y\rangle. \tag{7}$$

Indeed, since $\Phi(x)$ is closed and convex, there exists a unique element $\bar{y} \in \Phi(x)$ such that $d_\Phi(x) = \| \bar{y} \|$ and

$$\langle \bar{y}, y\rangle \geq \langle \bar{y}, \bar{y}\rangle, \forall y \in \Phi(x). \tag{8}$$

For every $y^* \in -C' \cap B_{\mathbb{R}^m}$, we have

$$-\sup_{y \in \Phi(x)} \langle y^*, y\rangle = \inf_{y \in \Phi(x)} \langle -y^*, y\rangle \leq \langle -y^*, \bar{y}\rangle \leq \| \bar{y} \| = d_\Phi(x).$$

Therefore,

$$d_\Phi(x) \geq \max_{y^* \in -C' \cap B_{\mathbb{R}^m}} - \sup_{y \in \Phi(x)} \langle y^*, y\rangle. \tag{9}$$

If $0 \in \Phi(x)$, then by choosing $y^* = 0 \in -C' \cap B_{\mathbb{R}^m}$, we have

$$d_\Phi(x) = -\sup_{y \in \Phi(x)} \langle y^*, y\rangle. \tag{10}$$

(9) and (10) imply (7).

If $0 \notin \Phi(x)$, then by taking (8) into account and choosing $\bar{y}^* = -\frac{\bar{y}}{\|\bar{y}\|} \in -C' \cap S_{\mathbb{R}^m}$, we have

$\langle \bar{y}^*, y\rangle \leq \langle \bar{y}^*, \bar{y}\rangle = -\| \bar{y} \|, \forall y \in \Phi(x)$.

Hence,

$$d_\Phi(x) = -\sup_{y \in \Phi(x)} \langle \bar{y}^*, y\rangle. \tag{11}$$

(9) and (11) imply (7).

For every $y^* \in -C' \cap B_{\mathbb{R}^m}$, one has

$$\sup_{y \in \Phi(x)} \langle y^*, y\rangle = \langle y^*, f(x) + a\rangle \tag{12}$$

which together with (7) gives

$$d_\Phi(x) = \max_{z^* \in C' \cap B_{\mathbb{R}^m}} \langle z^*, f(x) + a\rangle = [s(C' \cap B_{\mathbb{R}^m},.) \circ (f + a)](x).$$

Now, we consider the case when $d_\Phi(x) > 0$, or equivalently, $0 \notin \Phi(x)$. From (11) and (12), one has

$$d_\Phi(x) = -\sup_{y \in \Phi(x)} \langle \bar{y}^*, y\rangle = \langle z^*, f(x) + a\rangle$$

with $z^* = -\bar{y}^* = \frac{\bar{y}}{\|\bar{y}\|} \in C' \cap S_{\mathbb{R}^m}$. Suppose that we have $y^* \in C' \cap B_{\mathbb{R}^m}$ also satisfying

$$d_\Phi(x) = \langle y^*, f(x) + a\rangle = -\sup_{y \in \Phi(x)} \langle -y^*, y\rangle = \inf_{y \in \Phi(x)} \langle y^*, y\rangle.$$

Then,

$$\langle y^*, \bar{y}\rangle \geq d_\Phi(x) = \left\langle \frac{\bar{y}}{\| \bar{y} \|}, \bar{y}\right\rangle$$

which implies

$$\left\langle y^* - \frac{\bar{y}}{\| \bar{y} \|}, \bar{y}\right\rangle \geq 0.$$

pISSN 1859-1388
eISSN 2615-9678

Set $c = y^* - \frac{\bar{y}}{\|\bar{y}\|}$. We have

$$1 \geq \| y^* \|^2 = \| c + \frac{\bar{y}}{\| \bar{y} \|} \|^2 = \| c \|^2 + \| \frac{\bar{y}}{\| \bar{y} \|} \|^2 + 2 \left\langle c, \frac{\bar{y}}{\| \bar{y} \|} \right\rangle \geq 1 + \| c \|^2.$$

Hence, $c = 0$, which implies $y^* = z^*$.

Lemma 4.2 *Let $\bar{x} \in X$. If $d_\Phi(\bar{x}) > 0$ and f admits $A_f(\bar{x})$ as a bounded approximation at $\bar{x}$, then there exists $z^* \in C' \cap S_{\mathbb{R}^m}$ such that $z^* \circ A_f(\bar{x})$ is an approximation of d_Φ at $\bar{x}$, where $z^* \circ A_f(\bar{x}) := \{\langle z^*, \xi(.)\rangle : \xi \in A_f(\bar{x})\}$.*

Proof. By Lemma 4.1,

$d_\Phi = [s(C' \cap B_{\mathbb{R}^m}, .) \circ (f + a)]$

and there exists a unique element $z^* \in C' \cap B_{\mathbb{R}^m}$ satisfying

$s(C' \cap B_{\mathbb{R}^m}, f(x) + a) = \langle z^*, f(x) + a\rangle$.

Moreover, $\| z^* \| = 1$. By Proposition 4.1, the support function $s(C' \cap B_{\mathbb{R}^m}, .)$ is differentiable at $\bar{y} = f(\bar{x}) + a$ and

$\nabla s(C' \cap B_{\mathbb{R}^m}, .)(\bar{y}) = z^*$.

Hence, by Lemma 3.3 and Lemma 3.1, $z^* \circ A_f(\bar{x})$ is an approximation of d_Φ at $\bar{x}$.

Define

$\phi(x) := f(x) + C, \forall x \in X.$

Lemma 4.3 *If f is continuous, then so is d_ϕ.*

Proof. Suppose on the contrary that d_ϕ is not continuous. Then, there exist $\bar{x} \in X$, $\varepsilon > 0$ and a sequence $\{x_n\} \subset X$ such that

$$x_n \to \bar{x}$$
$$d_\phi(x_n) \notin d_\phi(\bar{x}) + (-\varepsilon, \varepsilon), \forall n.$$

Without loss of generality, we may assume one of the following cases holds.

i) $d_\phi(x_n) \leq d_\phi(\bar{x}) - \varepsilon, \forall n$. Since C is closed and convex, for every n, one can find a point $c_n \in C$ satisfying

$$f(x_n) + c_n \in B_{\mathbb{R}^m}(0, d_\phi(\bar{x}) - \varepsilon). \quad (13)$$

It can be verified that

$$\left[B_{\mathbb{R}^m}\left(f(\bar{x}), \frac{\varepsilon}{2}\right) + C\right] \cap B_{\mathbb{R}^m}(0, d_\phi(\bar{x}) - \varepsilon) = \emptyset.$$

By the continuity of f, there exists x_n such that

$$f(x_n) \in B_{\mathbb{R}^m}\left(f(\bar{x}), \frac{\varepsilon}{2}\right).$$

Hence,

$$f(x_n) + c_n \notin B_{\mathbb{R}^m}(0, d_\phi(\bar{x}) - \varepsilon)$$

which contradicts (13).

ii) $d_\phi(\bar{x}) \leq d_\phi(x_n) - \varepsilon, \forall n$. Analogously, we also get a contradiction. The lemma is proved.

Assumption 1:

$\exists \varepsilon > 0: \bigcup_{x \in B_X(\bar{x}, \varepsilon)} coA_f(x)$ is relatively compact.

Assumption 2: $\mathrm{co}A_f$ is closed at $\bar{x}$, i.e., if $x_n \to \bar{x}, w_n^* \to w^*$ with $w_n^* \in \mathrm{co}A_f(x_n)$, then $w^* \in \mathrm{co}A_f(\bar{x})$.

Theorem 4.1 *Assume that F has the Cl-property and f is continuous. Suppose that f admits an approximation $A_f(x)$ at every x in a neighborhood of $\bar{x} \in X$, which fulfills Assumptions 1,2. If $\bar{x}$ is a local efficient solution of (P)and (P) is regular at $\bar{x}$, then there exist $z^* \in C'\{0\}, y^* \in Y_F^*$ such that*

$$\begin{cases} 0 \in z^* \circ \mathrm{co}A_f(\bar{x}) - \partial C_F(y^*, .)(\bar{x}) \\ C_F(y^*, \bar{x}) = 0. \end{cases}$$

Proof. Our proof is similar to the ones used in [9] and [2]. Since C is not a subspace, one can find $c \in C$ with $\| c \| = 1$ and

$$c \notin -C. \quad (14)$$

For every $x \in X$, define

$$\phi_n(x) := f(x) - f(\bar{x}) + \frac{1}{n}c + C.$$
$$h_n(x) := \max\{d_{\phi_n}(x), d_F(x)\}.$$

We see that $h_n(\bar{x}) \leq \frac{1}{n} + \inf_{x \in X} h_n(x)$. By Lemma 4.3, d_{ϕ_n} is continuous and hence, h_n is too. Then, by Ekeland's Variational Principle, one can find $x_n \in X$ such that

$$\begin{cases} \| x_n - \bar{x} \| \leq \frac{1}{\sqrt{n}} \\ h_n(x_n) \leq h_n(x) + \frac{1}{\sqrt{n}} \| x_n - x \|, \forall x \in X. \end{cases}$$

Therefore, x_n is a minimum of the function $h_n(x) + \frac{1}{\sqrt{n}} \| x_n - x \|$. By Lemma 3.5 and Lemma 3.3,

$$0 \in clco\left[A_{h_n}(x_n) + \frac{1}{\sqrt{n}} B_{X^*}\right].$$

Taking Lemma 3.4 into account, we can see there exist $w_n \in X^*, \lambda_n \in [0,1]$ such that $w_n \to 0$ and

$$w_n \in \lambda_n coA_{d_{\phi n}}(x_n) + (1-\lambda_n)\,\partial d_F(x_n) + \frac{1}{\sqrt{n}} B_{X^*},$$

where $\lambda_n = 0$ if $d_F(x_n) > d_{\phi n}(x_n)$ and $\lambda_n = 1$ if $d_F(x_n) < d_{\phi n(x_n)}$.

Note that $h_n(x_n) > 0$; otherwise, $f(x_n) - f(\bar{x}) + \frac{1}{n} c \in -C, d_F(x_n) = 0$. Then, $0 \in F(x_n)$ (since $F(x_n)$ is closed) and by the assumption on $\bar{x}$, $f(\bar{x}) - f(x_n) \in -C$. This implies $c \in -C$, which contradicts (14).

By Lemma 2.2 and Lemma 4.2, there exist $z_n^* \in C' \cap S_{\mathbb{R}^m}, y_n^* \in Y_F^* \cap (S_{Y^*} \cup \{0\})$ such that

$$w_n \in \lambda_n z_n^* \circ coA_f(x_n) - (1-\lambda_n)\,\partial C_F(y_n^*, x_n) + \frac{1}{\sqrt{n}} B_{X^*}. \tag{15}$$

$$d_F(x_n) = -C_F(y_n^*, x_n). \tag{16}$$

We may assume that $\lambda_n \to \lambda \in [0,1], z_n^* \to z_0^* \in C' \cap S_{\mathbb{R}^m}$ and that

$$y_n^* \overset{w^*}{\to} y_0^* \in Y_F^* \tag{17}$$

(since $\{y_n^*\}$ is bounded and Y_F^* is weak *-closed). Let $n \to \infty$, by Assumptions 1,2 and by the Cl-property of F, we have

$$0 \in z^* \circ coA_f(\bar{x}) - \partial C_F(y^*, \bar{x}),$$

where

$$z^* = \lambda z_0^* \in C', y^* = (1-\lambda) y_0^* \in Y_F^*. \tag{18}$$

We shall show that $\lambda > 0$, then consequently, $z^* \neq 0$. Indeed, from (15), for every n , we can find $x_{1n}^* \in coA_f(x_n), x_{2n}^* \in \partial C_F(y_n^*, x_n), x_{3n}^* \in B_{X^*}$ satisfying

$$w_n = \lambda_n z_n^* x_{1n}^* - (1-\lambda_n) x_{2n}^* + \frac{1}{\sqrt{n}} x_{3n}^*. \tag{19}$$

By regularity, for n being sufficiently large, there exists $\xi_n \in \delta B_X$ such that

$$C_F(y_n^*, x_n) + \langle x_{2n}^*, \xi_n \rangle \geq \gamma. \tag{20}$$

(19) and (20) imply

$$\langle \lambda_n z_n^* x_{1n}^* + \frac{1}{\sqrt{n}} x_{3n}^* - w_n, \xi_n \rangle \geq (1-\lambda_n)(\gamma - C_F(y_n^*, x_n)).$$

Taking (16) into account, we have

$$\langle \lambda_n z_n^* x_{1n}^* - w_n, \xi_n \rangle + \frac{1}{\sqrt{n}} \delta \geq (1-\lambda_n)(\gamma + d_F(x_n)). \tag{21}$$

By Assumption 1, $\exists \eta > 0$ such that $\| x_{1n}^* \| \leq \eta$, for n being sufficiently large. Then from (21), one has

$$\lambda_n \eta \delta \geq \langle w_n, \xi_n \rangle - \frac{1}{\sqrt{n}} \delta + (1-\lambda_n)(\gamma + d_F(x_n)).$$

Since $\lim_{n\to\infty} d_F(x_n) = d_F(\bar{x}) = 0$, letting $n \to \infty$ gives

$$\lambda \eta \delta \geq (1-\lambda)\gamma,$$

which implies

$$\lambda \geq \frac{\gamma}{\eta\delta + \gamma} > 0.$$

Finally, by combining Lemma 2.3, Lemma 2.4, and (16) we have

$$C_F(y^*, \bar{x}) \leq \limsup_{n\to\infty} C_F((1-\lambda_n) y_n^*, \bar{x}) \leq \lim_{n\to\infty} (-(1-\lambda_n) d_F(x_n)) = 0.$$

Since $0 \in F(\bar{x})$, the converse inequality is obvious. Hence, $C_F(y^*, \bar{x}) = 0$. The proof is complete.

A special case of Theorem 4.1 when Y is finitely dimensional is remarkable.

pISSN 1859-1388
eISSN 2615-9678

Theorem 4.2 *Suppose that the assumptions of Theorem 4.1 are fulfilled and that Y is finitely dimensional. If $\bar{x}$ is a local efficient solution of (P), then there exist $z^* \in C', y^* \in Y_F^*$ with $(z^*, y^*) \neq (0,0)$ such that*

$$\begin{cases} 0 \in z^* \circ \text{co}A_f(\bar{x}) - \partial C_F(y^*, .)(\bar{x}) \\ C_F(y^*, \bar{x}) = 0. \end{cases}$$

If, in addition, (P) is regular at $\bar{x}$, then $z^* \neq 0$.

Proof. The proof is the same as the one of Theorem 4.1 with some notices as follows.

+ Expression (17)

$$y_n^* \overset{w^*}{\to} y_0^* \in Y_F^*$$

in the proof of Theorem 4.1 is replaced by

$$y_n^* \to y_0^* \in Y_F^* \cap (S_{Y^*} \cup \{0\})$$

since $y_n^* \in Y_F^* \cap (S_{Y^*} \cup \{0\})$ and Y^* is finitely dimensional.

+ If $y_0^* = 0$, then $\lambda = 1$.

+ Then from equalities (18)

$$z^* = \lambda z_0^* \in C', y^* = (1-\lambda) y_0^* \in Y_F^*$$

(where $z_0^* \in S_{\mathbb{R}^m}$), we deduce $(z^*, y^*) \neq (0,0)$.

We now present some applications of Theorem 4.2 to non-Lipschitz vector problems with constraints (2) or (3). Firstly, consider the following problem

$$\min f(x) \text{s.} t. g(x) \in K, (P')$$

where $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^n \to \mathbb{R}^l$ is a locally Lipschitz vector function and $K \subset \mathbb{R}^l$ is a closed convex cone. Set

$$F(x) := g(x) - K, \forall x \in \mathbb{R}^n.$$

We can verified that

$$Y_F^* = K'$$

$$C_F(y^*, x) = \langle y^*, g(x) \rangle, \forall x \in \mathbb{R}^n, y^* \in K',$$

where K' is the polar cone of K. Since $\partial C_F(y^*, x) = y^* \circ \partial g(x)$, F has the Cl-property. Then, we have the following result immediately from Theorem 4.1.

Corollary 4.1 *Assume that f is continuous. Suppose that f admits an approximation $A_f(x)$ at every x in a neighborhood of $\bar{x} \in X$ such that Assumptions 1,2 are fulfilled. If $\bar{x}$ is a local efficient solution of (P'), then there exist $z^* \in C', y^* \in K'$ with $(z^*, y^*) \neq (0,0)$ such that*

$$\begin{cases} 0 \in z^* \circ \text{co}A_f(\bar{x}) - y^* \circ \partial g(\bar{x}) \\ \langle y^*, g(\bar{x}) \rangle = 0. \end{cases}$$

If, in addition, (P′) is regular at $\bar{x}$, then $z^* \neq 0$.

Next, we consider the following problem

$$\min f(x) \text{s.} t. \begin{pmatrix} g_i(x) \leq 0, i = 1, \ldots, n \\ h_j(x) = 0, j = 1, \ldots, k, \end{pmatrix} \text{(P}''\text{)}$$

where $f: \mathbb{R}^l \to \mathbb{R}^m$ is a non-Lipschitz vector function, and $g_i, h_j: X \to \mathbb{R}$ are locally Lipschitz functions. Set

$$K := \mathbb{R}_+^n \times \{0_k\}$$
$$\phi(x) := (g_1(x), \ldots, g_n(x), h_1(x), \ldots, h_k(x))$$
$$F(x) := \phi(x) + K,$$

where $\mathbb{R}_+^n$ is the nonnegative orthant cone of $\mathbb{R}^n$, 0_k is the origin of $\mathbb{R}^k$. Then, ϕ, F are locally Lipschitz. We can see that the inclusion constraint $0 \in F(x)$ is equivalent to the system of equality/inequality constraints

$$g_i(x) \leq 0, i = 1, \ldots, n$$

$$h_j(x) = 0, j = 1, \ldots, k.$$

We have

$$Y_F^* = -K' = (-\mathbb{R}_+^n) \times \mathbb{R}^k$$

$$C_F(y^*, x) = \langle y^*, \phi(x) \rangle, \forall y^* \in Y_F^*$$

$$\partial C_F(y^*, x) = y^* \circ \partial \phi(x) \subset \sum_{i=1}^{n} y_i^* \partial g_i(x) + \sum_{j=1}^{k} y_{n+j}^* \partial h_j(\bar{x}), \forall y^* \in Y_F^*, x \in \mathbb{R}^l,$$

where $y^* = (y_1^*, \ldots, y_{n+k}^*)$. Since the Clarke generalized Jacobian $\partial \phi$ is closed at any point, it can be verified that F has the Cl-property. For every feasible solution x of Problem (P″) and $y^* \in Y_F^*$, we have

$$C_F(y^*, x) = 0 \Leftrightarrow \langle y^*, \phi(x)\rangle = 0 \Leftrightarrow \sum_{i=1}^{n} y_i^* g_i(\bar{x}) = 0.$$

Then, the following corollary is immediate from Theorem 4.2.

Corollary 4.2 *Assume that f is continuous. Suppose that f admits an approximation $A_f(x)$ at every x in a neighborhood of $\bar{x} \in X$ such that Assumptions 1,2 are fulfilled. If $\bar{x}$ is a local efficient solution of (P''), then there exist $z^* \in C', \lambda_1, \dots, \lambda_n \geq 0, \mu_1 \dots, \mu_k \in \mathbb{R}$ not all zero such that*

$$\begin{cases} 0 \in z^* \circ coA_f(\bar{x}) + \sum_{i=1}^{n} \lambda_i \, \partial g_i(\bar{x}) + \sum_{j=1}^{k} \mu_j \, \partial h_j(\bar{x}), \\ \sum_{i=1}^{n} \lambda_i g_i(\bar{x}) = 0. \end{cases}$$

If, in addition, (P'') is regular at $\bar{x}$, then $z^* \neq 0$.

If f is locally Lipschitz, then the Clarke generalized Jacobian $\partial f(x)$ is also an approximation and Assumptions 1,2 are satisfied. Then, from Corollary 4.2, we have

Corollary 4.3 *Assume that f is locally Lipschitz. If $\bar{x}$ is a local efficient solution of (P''), then there exist $z^* \in C', \lambda_1, \dots, \lambda_n \geq 0, \mu_1 \dots, \mu_k \in \mathbb{R}$ not all zero such that*

$$\begin{cases} 0 \in z^* \circ \partial f(\bar{x}) + \sum_{i=1}^{n} \lambda_i \, \partial g_i(\bar{x}) + \sum_{j=1}^{k} \mu_j \, \partial h_j(\bar{x}), \\ \sum_{i=1}^{n} \lambda_i g_i(\bar{x}) = 0. \end{cases}$$

If, in addition, (P'') is regular at $\bar{x}$, then $z^* \neq 0$.

Example 4.1 *Let H be a Hilbert space with a countable base $\{e_i : i = 1,2,..\}$ satisfying*

$$\langle e_i, e_j\rangle = \begin{cases} 1, i = j \\ 0, i \neq j. \end{cases}$$

For $a \in H$, set

$$\phi_a(x) := \langle a, x\rangle, \forall x \in H.$$

Define functions $f = (f_1, f_2): H \to \mathbb{R}^2, g: H \to \mathbb{R}$ as follows. For every $x = \sum_{i=1}^{\infty} t_i e_i$,

$$f_1(x) := \begin{cases} |t_1|, & t_1 \leq 1 \\ 1 + \sqrt{t_1 - 1}, & 1 < t_1 \leq 2 \\ 1 + \frac{t_1}{2}, & t_1 > 2 \end{cases}$$

$$f_2(x) := t_1$$

$$g(x) := \| x \|^2 - 1.$$

Then, we can verify that

$$A_{f_1}(x) := \begin{cases} \{\phi_{e_1}\}, & 0 < t_1 < 1 \\ \{t\phi_{e_1} : t \in [0, +\infty)\}, & t_1 = 1 \\ \{\frac{1}{2\sqrt{t_1 - 1}} \phi_{e_1}\}, & 1 < t_1 \leq 2 \\ \{\frac{1}{2} \phi_{t_1}\}, & t_1 > 2 \\ \{-\phi_{e_1}\}, & t_1 < 0 \\ \{t\phi_{e_1} : t \in [-1,1]\}, & t_1 = 0, \end{cases}$$

$$A_{f_2}(x) := \{\phi_{e_1}\}$$

are approximations of f_1, f_2 at x, respectively. Hence, by Lemma 3.1, we have

$$A_f(x) = A_{f_1}(x) \times A_{f_2}(x)$$

$$= \begin{cases} \{(\phi_{e_1}, \phi_{e_1})\}, & 0 < t_1 < 1 \\ \{(t\phi_{e_1}, \phi_{e_1}) : t \in [0, +\infty)\}, & t_1 = 1 \\ \{(\frac{1}{2\sqrt{t_1 - 1}} \phi_{e_1}, \phi_{e_1})\}, & 1 < t_1 \leq 2 \\ \{(\frac{1}{2} \phi_{t_1}, \phi_{e_1})\}, & t_1 > 2 \\ \{(-\phi_{e_1}, \phi_{e_1})\}, & t_1 < 0 \\ \{(t\phi_{e_1}, \phi_{e_1}) : t \in [-1,1]\}, & t_1 = 0 \end{cases}$$

which is an approximation of f at x. We see that Assumptions 1,2 are fulfilled at every $x = \sum_{i=1}^{\infty} t_i e_i$ with $t_1 \neq 1$. The function g is locally Lipschitz and differentiable with the derivative $Dg(x) = 2\phi_x$.

Assume that $\mathbb{R}^2$ is ordered by the cone $\mathbb{R}_+^2$. We consider the problem

$$\min f(x) \text{s.t.} \, g(x) \leq 0. \, (\text{P}_1)$$

Noting that $F(x) := g(x) + \mathbb{R}_+$ has the Cl-property and (P_1) is regular at every x and solving the system

$$\exists z^* \in \mathbb{R}_+^2 \backslash \{0\}, \exists \lambda \geq 0 : \begin{cases} g(x) \leq 0 \\ 0 \in z^* \circ coA_f(x) + \lambda Dg(x) \\ \lambda g(x) = 0 \end{cases}$$

pISSN 1859-1388
eISSN 2615-9678

we obtain the solution set

$$S = \{x = \sum_{i=1}^{\infty} t_i e_i : t_1 \leq 0\} \cap B_H(0,1)$$

which contains all candidates for a local efficient solution of Problem P_1. Moreover, by computing, we see that actually S coincides with the set of all local efficient solution of Problem P_1.

Funding statement

This research was supported by the Ministry of Education and Training of Vietnam under Grant B2019-DHH-11 (QD 5652-28/12/2018)

Conflict of interest

The author declares that he has no potential conflict of interest.

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