Hue University Journal of Science: Natural Science
Vol. 129, No. 1B, 79–86, 2020

pISSN 1859-1388
eISSN 2615-9678

# REPRESENTATION OF SOME SPECIAL FUNCTIONS ON TRANSCENDENCE BASIS

**Bui Van Chien***

University of Sciences, Hue University, 77 Nguyen Hue St., Hue, Vietnam

* Correspondence to Bui Van Chien <bvchien@hueuni.edu.vn>
*(Received: 06 January 2020; Accepted: 27 March 2020)*

**Abstract.** The special functions such as multiple harmonic sums, polyzetas, or multiple polylogarithms are compatible with the structure of quasi-shuffle algebras. We express non-commutative generating series of these special functions on the transcendence bases of the algebras and then identify local coordinates to reduce their polynomial relations or asymptotic expansions indexed by these bases.

**Keywords:** quasi-shuffle product, special functions, multiple harmonic sum, polyzetas, multiple polylogarithms.

## 1 Introduction

A harmonic sum for the simple index, $s \in \mathbb{N}_+$, is defined by the sum $\mathrm{H}_s(N) := 1 + \frac{1}{2^s} + \ldots + \frac{1}{N^s}$. We know that the limit $\lim_{N\to\infty} \mathrm{H}_s(N)$ is also finite whenever $s > 1$ and one calls this limit the zeta number. For example

$$\lim_{N\to\infty} \mathrm{H}_2(N) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

These definitions are also extended to a set of multi-index called multiple harmonic sums and polyzetas (or multiple zeta values), respectively. For each composition of positive integers $s = (s_1, \ldots, s_r), s_1 > 1, r, N \in \mathbb{N}_+$,

$$\mathrm{H}_s(N) \quad := \sum_{N\geq n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}, \tag{1}$$

$$\zeta(s) \quad := \sum_{n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{2}$$

**Example 1.**

$$\begin{aligned}
&\mathrm{H}_{2,1}(N) = \sum_{N\geq n_1>n_2\geq 1} \frac{1}{n_1^2 n_2} \\
&= \frac{1}{2^2.1} + \left(\frac{1}{3^2.2} + \frac{1}{3^2.1}\right) \\
&+ \left(\frac{1}{4^2.3} + \frac{1}{4^2.2} + \frac{1}{4^2.1}\right) + \ldots \\
&+ \left(\frac{1}{N^2(N-1)} + \frac{1}{N^2(N-2)} + \ldots + \frac{1}{N^2.1}\right) \\
&\xrightarrow{N\to\infty} \sum_{n_1>n_2\geq 1} \frac{1}{n_1^2 n_2} = \zeta(2,1).
\end{aligned}$$

Furthermore, this structure also has an other infinity form, called multiple polylogarithms, such a function of one variable in the open unit ball of the complex plan, $z \in \mathbb{C}, |z| < 1$,

$$\mathrm{Li}_s(z) \quad := \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{3}$$

They all have famous relations in limits by Abel's theorem:

$$\lim_{N\to\infty} \mathrm{H}_s(N) = \lim_{z\to 1} \mathrm{Li}_s(z) = \zeta(s), \quad \forall s_1 > 1. \tag{4}$$

Fortunately, the multiple harmonic sums are compatible with the algebra of the stuffle product; whereas, the multiple polylogarithms are compatible with the algebra of the shuffle product when they are observed in the forms of iterated Chen integrals. As a consequence of these results, the polyzetas are compatible with both of the structures.

In this paper, we briefly review a general result about Hopf algebras (in Section 2), of the quasi-shuffle product and the concatenation product, constructed on a space of formal polynomials freely generated by some alphabet. They admit transcendence bases (see [1]) on which the special functions can be expressed as non-commutative generating series in respect of Hausdorff group[1]:

$$\begin{aligned}
\mathrm{H}(N) &= \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} \exp(\mathrm{H}_{\Sigma_l}(N)\Pi_l), \\
\mathbb{Z}_{\text{\mbox{⧢}}} &= \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} \exp(\zeta(\Sigma_l)\Pi_l), \\
\mathrm{L}(z) &= \prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow} \exp(\mathrm{Li}_{S_l}(z)P_l), \\
\mathbb{Z}_{\sqcup\!\sqcup} &= \prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow} \exp(\zeta(S_l)P_l).
\end{aligned}$$

Thanks to relations among the non-commutative generating series $\mathrm{L}(z) \xrightarrow{z\to 1} Z_{\sqcup\!\sqcup}, \mathrm{H}(N) \xrightarrow{N\to\infty} \mathbb{Z}_{\text{⧢}}$ in

[1]Hausdorff group is the group of group-like elements in a Hopf algebra.
$\mathcal{L}ynX, \mathcal{L}ynY$ denote the sets of Lyndon words generated by the alphabets $X = \{x_0, x_1\}, x_0 \prec x_1$, and $Y = \{y_k\}_{k\geq 1}, y_1 \succ y_2 \succ \ldots$.

DOI: 10.26459/hueuni-jns.v129i1B.5636

the equivalent algebraic structures, we establish relations and reduce representations in the forms of polynomial relations and asymptotic expansions indexed by the transcendence bases.

## 2 Quasi-shuffle algebra with the deformation $q$

Let's denote $Y := \{y_k | \ k \in \mathbb{N}_+\}$ an alphabet totally ordered by $y_1 \succ y_2 \succ \cdots$. A word is a finite sequence of letters and $Y^*$ denotes the set of all words including the empty word, denoted by $1_{Y^*}$. This set is a free monoid[2] and $1_{Y^*}$ is a neutral element. We call each linear combination, over the field $\mathbb{Q}$, of words in $Y^*$ a (formal) polynomial and $\mathbb{Q}\langle Y\rangle$ denotes the set of all polynomials. This set equipped with the concatenation product follows a free algebra with unit $1_{Y^*}$. A Lyndon word is a nonempty word that is smaller than all its nontrivial proper right factors and $\mathcal{L}ynY$ denotes the set of all Lyndon words in $Y^*$.

For any $q$ belonging to any field containing the field of rational number, the $q-$stuffle product, denoted by $⧢_q$, is defined by recurrent formula as follows: $\forall y_{k_1}, y_{k_2} \in Y, \forall u, v \in Y^*$

$$\begin{aligned} u ⧢_q 1_{Y^*} &= 1_{Y^*} ⧢_q u = u, \\ y_{k_1}u ⧢_q y_{k_2}v &= y_{k_1}(u ⧢_q y_{k_2}v) + y_{k_2}(y_{k_1}u ⧢_q v) \\ &+ \ q y_{k_1+k_2}(u ⧢_q v). \end{aligned}$$

**Example 2.**

$$\begin{aligned} y_2 ⧢_q y_3y_1 &= y_2(1_{Y^*} ⧢_q y_3y_1) + y_3(y_2 ⧢_q y_1) \\ &+ \ q y_5(1_{Y^*} ⧢_q y_1) \\ &= y_2y_3y_1 + y_3y_2y_1 + y_3y_1y_2 \\ &+ \ q(y_3y_3 + y_5y_1). \end{aligned}$$

This product is exactly the shuffle product (denoted by $⧢$) for $q = 0$ and the stuffle product (denoted by $⧢$ ) for $q = 1$. This product is commutative and associative hence, $(\mathcal{A}\langle Y\rangle, ⧢_q, 1_{Y^*})$ is a commutative, associative algebra with unit, where $\mathcal{A} := \mathbb{Q}[q]$ is the field extension of $\mathbb{Q}$ containing $q$. Here, we still use the notation $⧢_q$ as a morphism

$$\begin{aligned} ⧢_q : \mathcal{A}\langle Y\rangle \otimes \mathcal{A}\langle Y\rangle &\longrightarrow \mathcal{A}\langle Y\rangle \\ u \otimes v &\longmapsto u ⧢_q v. \end{aligned} \tag{5}$$

We denote $\Delta_{⧢_q}$ and $\Delta_{conc}$ as the dual laws of the $q-$stuffle product and the concatenation product, respectively; this means that for all $w$ in $Y^*$,

$$\begin{aligned} \Delta_{⧢_q}(w) &= \sum_{u,v \in Y^*} \langle \Delta_{⧢_q}(w) \mid u \otimes v\rangle u \otimes v \\ &= \sum_{u,v \in Y^*} \langle w \mid u ⧢_q v\rangle u \otimes v, \end{aligned} \tag{6}$$

$$\begin{aligned} \Delta_{conc}(w) &= \sum_{u,v \in Y^*} \langle \Delta_{conc}(w) \mid u \otimes v\rangle u \otimes v \\ &= \sum_{u,v \in Y^*} \langle w \mid uv\rangle u \otimes v. \end{aligned} \tag{7}$$

We proved (in paper [1]) that the coproduct $\Delta_{⧢_q}$ is compatible with the concatenation product. This means that $\Delta_{⧢_q}(uv) = \Delta_{⧢_q}(u)\Delta_{⧢_q}(v)$, whereas $\Delta_{conc}$ is compatible with the $q$-stuffle product, that means $\Delta_{conc}(u ⧢_q v) = \Delta_{⧢_q}(u) ⧢_q \Delta_{⧢_q}(v)$. An important point to note here is the weight of the word $w = y_{s_1} \ldots y_{s_r}$ to be (and denoted by) $(w) = s_1 + \ldots + s_r$. Due to these definitions we can see that $\Delta_{⧢_q}(w)$ is the polynomial of words in weight $(w)$ and $u ⧢_q v$ is the polynomial of words in weight $(u) + (v)$. Consequently, they all form the two algebraic structures in duality as follows:

**Proposition 1** ([1]). *$(\mathcal{A}\langle Y\rangle, conc, 1_{Y^*}, \Delta_{⧢_q}, \varepsilon, \mathcal{S}_q^{conc})$ and $(\mathcal{A}\langle Y\rangle, ⧢_q, 1_{Y^*}, \Delta_{conc}, \varepsilon, \mathcal{S}^{⧢_q})$ are the graded Hopf algebras in duality.*

On the other hand, we proved that the algebraic morphism defined on letters by

$$\pi_1(y_k) = y_k + \sum_{i \geq 2} \frac{(-q)^{i-1}}{i} \sum_{s_1 + \ldots + s_i = k} y_{s_1} \ldots y_{s_i} \tag{8}$$

to be an isomorphism between the two algebraic algebras $\mathcal{H}_{⧢} = (\mathcal{A}\langle Y\rangle, conc, 1_{Y^*}, \Delta_{⧢}, \varepsilon, \mathcal{S}_q)$ and $\mathcal{H}_{⧢_q} = (\mathcal{A}\langle Y\rangle, conc, 1_{Y^*}, \Delta_{⧢_q}, \varepsilon, \mathcal{S}_q)$. Therefore, each letter is a primitive[3] element in $\mathcal{H}_{⧢}$ and follows its image $\pi_1(y_k)$ to be primitive in $\mathcal{H}_{⧢_q}$. This result helps us to construct a linear basis for the space of the Lie algebra generated by primitive elements. We denote here by $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ the Poincaré-Birkhoff Witt basis (PBW-basis for short), and it is computed according to the recurrent formula [1]:

$$\begin{cases} \Pi_{y_s} &= \pi_1(y_s) \quad \text{for } y_s \in Y, \\ \Pi_l &= [\Pi_{l_1}, \Pi_{l_2}] \quad \text{for } l \in \mathcal{L}ynY \setminus Y, \\ \Pi_w &= \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k} \quad \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \end{cases} \tag{9}$$

where $(l_1, l_2)$ is the standard factorization of $l$, $w = l_1^{i_1} \ldots l_k^{i_k}$, $l_1 > \ldots > l_k$, $l_1, \ldots, l_k \in \mathcal{L}ynY$,

**Example 3.**

$$\begin{aligned} \Pi_{y_1} &= y_1, \quad \Pi_{y_2} = y_2 - \tfrac{q}{2}y_1^2, \\ \Pi_{y_2y_1} &= y_2y_1 - y_1y_2, \\ \Pi_{y_3y_1y_2} &= y_3y_1y_2 - y_2y_3y_1 + y_2y_1y_3 - \tfrac{q}{2}y_3y_1^3 \\ &- \ q y_2y_1^2y_2 - y_1y_3y_2 + \tfrac{q}{2}y_1y_3y_1^2 \\ &+ \ \tfrac{q}{2}y_1^2y_2^2 - \tfrac{q^2}{2}y_1^2y_2y_1^2 + \tfrac{q}{2}y_2^2y_1^2 \\ &+ \ \tfrac{q}{2}y_1^2y_3y_1 - \tfrac{q}{2}y_1^3y_3 + \tfrac{q^2}{4}y_1^4y_2 + \tfrac{q^2}{4}y_2y_1^4. \end{aligned}$$

On the other hand, we also established a formula for the dual basis[4], denoted by $\{\Sigma_w\}_{w \in Y^*}$, by

[2]The binary operation here is the concatenation product.
[3]A polynomial $P$ is primitive for the coproduct $\Delta_{⧢}$ if $\Delta_{⧢}(P) = P \otimes 1_{Y^*} + 1_{Y^*} \otimes P$.
[4]This pair of bases is dual in meaning that $\langle \Sigma_u \mid \Pi_v\rangle = \delta_{u,v}$ for any $u, v \in Y^*$.

 pISSN 1859-1388
eISSN 2615-9678

the recurrent formula [1]:

$$\begin{cases} \Sigma_{y_k} &= y_k, \\ \Sigma_l &= \sum_{(*)_1} y_{s_1} \Sigma_{l_1 \ldots l_n} \\ &+ \sum_{i \geq 2} \frac{q^{i-1}}{i!} \sum_{(*)_2} y_{s'_1 + \ldots + s'_i} \Sigma_{l_1 \ldots l_n}, \\ \Sigma_w &= \frac{\Sigma_{l_1}^{⧢_q i_1} ⧢_q \ldots ⧢_q \Sigma_{l_k}^{⧢_q i_k}}{i_1! \ldots i_k!}. \end{cases} \quad (10)$$

**Example 4.**

$$\begin{aligned} \Sigma_{y_1} &= y_1, \quad \Sigma_{y_2} = y_2, \quad \Sigma_{y_3} = y_3, \\ \Sigma_{y_3 y_1 y_2} &= y_3 y_1 y_2 + y_3 y_2 y_1 \\ &+ q(y_3^2 + \tfrac{1}{2} y_4 y_2 + \tfrac{1}{2} y_5 y_1) + \tfrac{q^2}{3} y_6. \end{aligned}$$

This basis reduces a transcendence basis, $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$, of the algebra $(\mathcal{A}\langle Y \rangle, ⧢_q, 1_{Y^*})$. This permits us to express the diagonal series $\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w$, an element in the algebra $\mathcal{A}\langle\langle Y \rangle\rangle \otimes \mathcal{A}\langle\langle Y \rangle\rangle$ of the $q$-stuffle product on the left of the tensor and the concatenation product on the right.

**Proposition 2** ([1]).

$$\begin{aligned} \mathcal{D}_Y &= \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w \\ &= \sum_{l_1 \succ \ldots \succ l_k} \frac{\Sigma_{l_1}^{⧢_q i_1} ⧢_q \ldots ⧢_q \Sigma_{l_k}^{⧢_q i_k}}{i_1! \ldots i_k!} \otimes \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k} \\ &= \prod_{l \in \mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l), \end{aligned} \quad (11)$$

*where the last product takes Lyndon words in decreasing order.*

## 3 Representation of special functions on transcendence bases

### 3.1 Representation of multiple polylogarithms

We now consider the above algebra in the case of the alphabet $X = \{x_0, x_1\}$, totally ordered by $x_0 \prec x_1$, with the shuffle product (it means $q = 0$). At that time, the couple of bases in duality [2] is denoted by $\{P_w\}_{w \in X^*}$, the PBW-basis, and $\{S_w\}_{w \in X^*}$, Schützenberger basis. It follows from (11) that[5]

$$\begin{aligned} \mathcal{D}_X &:= \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w \\ &= \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l). \end{aligned} \quad (12)$$

We have seen at (3) that a multiple polylogarithms is determined for each multi-index $s = (s_1, \ldots, s_r)$. In this section, we use encoding that each composition of positive integers $s = (s_1, \ldots, s_r)$ associates with the word $w = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1$. Thus, the multiple polylogarithms can be rewritten as:

$$\mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r \geq 1} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \quad |z| < 1. \quad (13)$$

Using two differential forms $\omega_0(z) := \frac{dz}{z}$ and $\omega_1(z) := \frac{dz}{1-z}$ with the conventions that $\mathrm{Li}_{1_X^*} = 1$ and $\mathrm{Li}_{x_0}(z) = \int_1^z \omega_0(t) = \log(z)$, one can express the multiple polylogarithms, thank to Frederich criterion, in the form of iterated integral [3, 4],

$$\begin{aligned} \mathrm{Li}_{x_1}(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \omega_1(t) = -\log(1-z), \\ \mathrm{Li}_{x_i w}(z) &= \int_0^z \omega_i \, \mathrm{Li}_w, \text{ for } i \in \{0, 1\}. \end{aligned} \quad (14)$$

Following this representation, one proved that the multiple polylogarithms are compatible with the shuffle product, namely [2, 5]:

$$\forall u, v \in X^*, \quad \mathrm{Li}_u(z) \, \mathrm{Li}_v(z) = \mathrm{Li}_{u ⧢ v}(z). \quad (15)$$

This permits us to extend Li as a morphism:

**Theorem 1** ([3]). *Let $\mathcal{C} := \mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}\right]$. The mapping $w \longmapsto \mathrm{Li}_w$ is the isomorphism of $(\mathcal{C}\langle X \rangle, ⧢, 1_{X^*})$ to $(\mathcal{C}[\{\mathrm{Li}_w\}_{w \in X^*}], \cdot, 1_\Omega)$, where $\Omega := \mathbb{C} \setminus ((-\infty, 0) \cup [1, +\infty))$.*

The non-commutative generating series of multiple polylogarithms is defined as an image of the morphism[6] on the double series $\mathcal{D}_X$ though $\mathrm{Li}_\bullet \otimes id_{X^*}$:

$$\begin{aligned} \mathrm{L}(z) &:= \mathrm{Li}_\bullet \hat{\otimes} id_{X^*}(\mathcal{D}_X) = \sum_{w \in X^*} \mathrm{Li}_w(z) w \\ &= e^{-\log(1-z) x_1} \prod_{l \in (\mathcal{L}ynX) \setminus X}^{\searrow} e^{\mathrm{Li}_{S_l}(z) P_l} e^{\log(z) x_0}. \end{aligned}$$

On the other hand, for any Lyndon word $l \in (\mathcal{L}ynX) \setminus X$, one has $S_l \in x_0 X^* x_1$. Therefore, we can consider the non-commutative generating series, denoted by $\mathrm{L}_{reg}$, as well as its evaluation at $z = 1$ [3], we have

$$\begin{aligned} \mathrm{L}_{reg}(z) &= \prod_{l \in (\mathcal{L}ynX) \setminus X}^{\searrow} e^{\mathrm{Li}_{S_l}(z) P_l}, \\ \xrightarrow{z \to 1} Z_⧢ &:= \prod_{l \in (\mathcal{L}ynX) \setminus X}^{\searrow} e^{\zeta(S_l) P_l}. \end{aligned} \quad (16)$$

Moreover, one studies about the monodromy of the multiple polylogarithms on close curves by Chen's series and the differential equation Drinfield [3, 6] to state the following proposition:

[5]Note that, $x_0, x_1$ are respectively the smallest and the largest Lyndon words in $X^*$ and $P_{x_0} = S_{x_0} = x_0, P_{x_1} = S_{x_1} = x_1$.

[6]This morphism isn't continue on the tensor product $(\mathbb{Q}\langle X \rangle, ⧢, 1_{X^*}) \otimes (\mathbb{Q}\langle X \rangle, conc, 1_{X^*})$ but in the subalgebra $Iso_{\mathbb{Q}\langle X \rangle} = span_{\mathbb{Q}}\{u \otimes v \mid |u| = |v|\}$.

**Proposition 3** ([3]). *i) For all curve $z_0 \rightsquigarrow z$ in $\Omega$, one has $\mathrm{L}(z) = S_{z_0 \rightsquigarrow z}\mathrm{L}(z_0)$.*

*ii) In the special case of curve $1-t$, one has*[7]

$$\mathrm{L}(x_0, x_1 \mid 1-t) = \mathrm{L}(-x_1, -x_0 \mid t) Z_{\sqcup\!\sqcup}. \tag{17}$$

We now use an automorphism of $\mathbb{Q}\langle X\rangle$ of the concatenation product, denoted by $\sigma$, verified $\sigma(x_0) = -x_1, \sigma(x_1) = -x_0$. Note that, for all words $w \in X^*$, $P_w$ and $S_w$ are homogeneous polynomial of weight[8] $|w|$, the length of $w$. Furthermore, $\mathbb{Q}\langle X\rangle$ is a graded space admitting two graded bases $\{P_w\}_{w\in X^*}$ and $\{S_w\}_{w\in X^*}$. We can see more precise by the following diagram illustrating a matrix representation of $\sigma$ in a subspace of weight $n$, denoted by $X_n := span\{u_1^{(n)}, \ldots, u_{2^n}^{(n)}\}$, where $\{u_1^{(n)}, \ldots, u_{2^n}^{(n)}\}$ is the set of all words of weight (the length in this case) $n$, $E^{(n)}$ denotes the matrix representation of $\sigma$ with respect to this basis: for all $1 \le i, j \le 2^n$,

$$E_{ij}^{(n)} := \langle \sigma(P_{u_i^{(n)}}) \mid P_{u_j^{(n)}} \rangle = \langle \sigma(S_{u_j^{(n)}}) \mid S_{u_i^{(n)}} \rangle, \tag{18}$$

$$\begin{tikzcd}
(X_n, \{P_{u_i^{(n)}}\}_{1\le i\le 2^n}) \arrow[r, "\sigma", "E^{(n)}"'] \arrow[d, dotted, "duality"'] & (X_n, \{P_{u_i^{(n)}}\}_{1\le i\le 2^n}) \arrow[d, dotted, "duality"] \\
(X_n, \{S_{u_i^{(n)}}\}_{1\le i\le 2^n}) & (X_n, \{S_{u_i^{(n)}}\}_{1\le i\le 2^n}). \arrow[l, "\sigma"', "{}^tE^{(n)}"]
\end{tikzcd}$$

**Proposition 4.** *Let $\mathrm{L}(z)$ be the non-commutative generating series of multiple polylogarithms, we have*

$$\sigma[\mathrm{L}(z)] = \sum_{w\in X^*} \mathrm{Li}_{S_w}(z)\sigma(P_w) = \sum_{w\in X^*} \mathrm{Li}_{\sigma(S_w)}(z) P_w. \tag{19}$$

*Proof.*

$$\begin{aligned}
\sum_{w\in X^*} \mathrm{Li}_{S_w}(z)\sigma(P_w) &= \sum_{n\ge 0}\sum_{i=1}^{2^n} \mathrm{Li}_{S_{u_i}}(z)\sigma(P_{u_i}) \\
&= \sum_{n\ge 0}\sum_{i=1}^{2^n} \mathrm{Li}_{S_{u_i}}(z) \sum_{j=1}^{2^n} E_{ij}^{(n)} P_{u_j} \\
&= \sum_{n\ge 0}\sum_{j=1}^{2^n}\sum_{j=1}^{2^n} E_{ij}^{(n)} \mathrm{Li}_{S_{u_i}}(z) P_{u_j} \\
&= \sum_{n\ge 0}\sum_{j=1}^{2^n} \mathrm{Li}_{\sum_{j=1}^{2^n} E_{ij}^{(n)} S_{u_i}}(z) P_{u_j} \\
&= \sum_{n\ge 0}\sum_{j=1}^{2^n} \mathrm{Li}_{\sigma(S_{u_j})}(z) P_{u_j} \\
&= \sum_{w\in X^*} \mathrm{Li}_{\sigma(S_w)}(z) P_w.
\end{aligned}$$

□

For this reason, we can rewrite relation (17) as follows:

$$\sum_{w\in X^*} \mathrm{Li}_{S_w}(z) P_w = \sum_{w\in X^*} \mathrm{Li}_{\sigma(S_w)}(1-z) P_w Z_{\sqcup\!\sqcup} \tag{20}$$

[7]Here, we understand $\mathrm{L}(z)$ as $\mathrm{L}(x_0, x_1|z)$.

[8]$|w|$ denotes the length of the word $w$.

From this formula, by identifying local coordinates, we get relations among the multiple polylogarithms indexed by basis $\{S_l\}_{l\in\mathcal{L}ynX}$. The following example are computed by our program running under Maple.

**Example 5.**

$$\begin{aligned}
\mathrm{Li}_{S_{x_0}}(z) &= \log(z), \quad \mathrm{Li}_{S_{x_1}}(z) = -\log(1-z), \\
\mathrm{Li}_{S_{x_0x_1}}(z) &= -\log(z)\log(1-z) \\
&- \mathrm{Li}_{S_{x_0x_1}}(1-z) + \zeta(S_{x_0x_1}), \\
\mathrm{Li}_{S_{x_0x_1^2}}(z) &= \frac{1}{2}\log(1-z)^2\log(z) \\
&+ \log(1-z)\,\mathrm{Li}_{S_{x_0x_1}}(1-z) \\
&- \mathrm{Li}_{S_{x_0^2x_1}}(1-z) \\
&+ \zeta(S_{x_0^2x_1}) + \log(z)\zeta(S_{x_0x_1}).
\end{aligned}$$

### 3.2 Representation of multiple harmonic sums

We have seen at (1) that a multiple harmonic sum is determined for each multi-index $s = (s_1, \ldots, s_r) \in \mathbb{N}_+^*$. Similar to the idea of the previous subsection, these compositions of positive integers, $s = (s_1, \ldots, s_r) \in Y^*$, are encoded by the words $w = y_{s_1} \ldots y_{s_r}$. Thus, the multiple harmonic sums can be rewritten as

$$\mathrm{H}_w(N) := \sum_{N\ge n_1>\ldots>n_r\ge 1} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{21}$$

Note that, for each composition $s = (s_1, s_2, \ldots, s_r)$, we have the reducing expression

$$\mathrm{H}_{y_s}(N) = \sum_{n_1=r}^{N} \frac{\mathrm{H}_{(y_{s_2},\ldots,y_{s_r})}(n_1-1)}{n_1}, \tag{22}$$

by the reason

$$\begin{aligned}
\mathrm{H}_{y_s}(N) &= \sum_{N\ge n_1>\ldots>n_r\ge 1} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}} \\
&= \sum_{n_1=r}^{N} \frac{1}{n_1^{s_1}} \sum_{n_1-1\ge n_2>\ldots>n_r\ge 1} \frac{1}{n_2^{s_2}\ldots n_r^{s_r}} \\
&= \sum_{n_1=r}^{N} \frac{\mathrm{H}_{(y_{s_2},\ldots,y_{s_r})}(n_1-1)}{n_1^{s_1}}.
\end{aligned}$$

This allows us to prove, by induction, that multiple harmonic sums are compatible with the stuffle product [7]. It means that for all words $w_1, w_2 \in Y^*$, we have

$$\mathrm{H}_{w_1}(N)\mathrm{H}_{w_2}(N) = \mathrm{H}_{w_1 \uplus w_2}(N). \tag{23}$$

**Proposition 5.** *The mapping $w \longmapsto H_w$ is the isomorphism between $(\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*})$ and the algebra of multiple harmonic sums with the standard product, denoted by $(\mathcal{H}_\mathbb{R}, \cdot, 1)$.*

 pISSN 1859-1388
eISSN 2615-9678

Because the set of the Lyndon words freely generates the algebra of the quasi-shuffle product [8], it follows the isomorphism $\mathcal{H}_{\mathbb{R}} \simeq \mathbb{Q}[\mathrm{H}_l, l \in \mathcal{L}ynY]$. Moreover, by using the expression of diagonal series $\mathcal{D}_Y$ (see (11)), we can factorize the non-commutative generating series of multiple harmonic sums $\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w$ as follows

**Proposition 6.**

$$\mathrm{H} = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\mathrm{H}_{\Sigma_l}\Pi_l) \quad (24)$$

$$= \exp \mathrm{H}_{y_1} y_1 \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} \exp(\mathrm{H}_{\Sigma_l}\Pi_l). \quad (25)$$

The original generating series of multiple harmonic sums forms a multiple polylogarithms deformed the factor $\frac{1}{1-z}$, namely for all multiindices $s = (s_1, s_2, \ldots, s_r)$,

$$\sum_{n\geq 0} \mathrm{H}_s(n) z^n = \frac{\mathrm{Li}_s(z)}{1-z}. \quad (26)$$

Indeed,

$$\begin{aligned} \frac{\mathrm{Li}_s(z)}{1-z} &= \sum_{n\geq 0} z^n \sum_{n_1>\ldots>n_r\geq 1} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \\ &= \sum_{n\geq r} z^n \sum_{n\geq n_1>\ldots>n_r\geq 1} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}} \\ &= \sum_{n\geq 0} \mathrm{H}_s(n) z^n. \end{aligned}$$

Here we accept that $\mathrm{H}_s(n) = 0$ for any $n < r$. In other words, $\mathrm{H}_s(N)$ is the coefficient of $z^N$ in the Taylor development of $\frac{\mathrm{Li}_s(z)}{1-z}$ in the system $\{z^N | N \in \mathbb{N}\}$. By the way, according to the representations of multiple polylogarithms (in the above subsection) we obtain relations or asymptotic expansions of multiple harmonic sums.

### 3.2.1 Generating series of multiple harmonic sums on the alphabet $X$

For any word $w \in X^*$, we denote $\mathrm{G}_w^X(z) := \frac{\mathrm{L}_w(z)}{1-z}$ and $\mathrm{G}^X(z) := \sum_{w\in X^*} \mathrm{G}_w^X(z)$. By the way, using formula (20), we have the following expressions:

$$\mathrm{G}^X(1-z) = \frac{1-z}{z} \sigma[\mathrm{G}^X(z)] Z_{\sqcup\!\sqcup},$$

$$\sum_{w\in X^*} \mathrm{G}_w^X(z) = \frac{1-z}{z} \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\mathrm{Li}_{S_l}(z)\sigma(P_l)) Z_{\sqcup\!\sqcup}.$$

**Example 6.** According to equality (27), we reduce the following relations by identifying local coordinates[9]:

$$\mathrm{G}_{S_{x_0}}^X(1-z) = \frac{\log(1-z)}{z},$$

$$\mathrm{G}_{S_{x_1}}^X(1-z) = -\frac{\log(z)}{z},$$

$$\mathrm{G}_{S_{x_0x_1}}^X(1-z) = \frac{z-1}{z}\mathrm{G}_{S_{x_0x_1}}^X(z)$$
$$+\frac{1}{z}\log z \log\frac{1}{1-z} + \zeta(S_{x_0x_1}),$$

$$\mathrm{G}_{S_{x_0^2x_1}}^X(z) = \frac{z-1}{z}(-\mathrm{G}_{S_{x_0^2x_1}}^X(z)$$
$$+\log z \mathrm{G}_{S_{x_0x_1}}^X(z) + \frac{\log^2 z\log(1-z)}{1-z} + \frac{\zeta(S_{x_0^2x_1})}{1-z}),$$

$$\mathrm{G}_{S_{x_0x_1^2}}^X(z) = \frac{1-z}{z}(-\mathrm{G}_{S_{x_0^2x_1}}^X(z)$$
$$+\log z \mathrm{G}_{S_{x_0x_1}}^X(z) - \log^2 z \mathrm{G}_{S_{x_1}}^X(z)$$
$$+\frac{\zeta(S_{x_0^2x_1})}{1-z}).$$

We use the notation $[z^N]\mathrm{G}_w^X(z)$ for the coefficient of $z^N$ in the Taylor development of $\mathrm{G}_w^X(z)$ in the scale of comparison $\{(1-z)^i \log^j(1-z), i, j \in \mathbb{N}\}$. From the representation of $\mathrm{G}_w^X(z)$, we can reduce asymptotic expansions of multiple harmonic sums in the scale of comparison $\{z^i \log^j(n), i, j \in \mathbb{N}\}$.

**Example 7.**

$$\begin{aligned} \mathrm{H}_{S_{x_0x_1^2}} &= [z^N]\mathrm{G}_{S_{x_0x_1^2}}^X \\ &= \zeta(S_{x_0x_1^2}) - \frac{\log N + 1 + \gamma}{N} + \frac{1}{2}\frac{\log N}{N^2} \\ &+ O(\frac{1}{N^3}). \end{aligned}$$

### 3.2.2 Generating series of multiple harmonic sums on the alphabet $Y$

We now use the linear projection $\pi_Y : \mathbb{Q}\langle X\rangle \longrightarrow \mathbb{Q}\langle Y\rangle$ which associates every word $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ with the word $y_{s_1}\ldots y_{s_r}$ and admits the convention $\pi_Y(wx_0) = 0$ for any $w \in X^*$. Then, for any word $w = y_{s_1}\ldots y_{s_r} \in Y^*$, we set $\mathrm{G}_w^Y(z) := \frac{\mathrm{L}_{(s_1,\ldots,s_r)}(z)}{1-z}$ and

$$\mathrm{G}^Y(z) := \pi_Y \mathrm{G}^X(z) = \pi_Y \sum_{w\in X^*} \mathrm{G}_w^X(z) w = \sum_{w\in Y^*} \mathrm{G}^Y(z) w.$$

From this and formula (20), we have

$$\mathrm{G}^Y(z) \underset{z\to 1}{\widetilde{\quad}} \exp((y_1+1)\log\frac{1}{1-z})\pi_Y Z_{\sqcup\!\sqcup}. \quad (27)$$

Moreover, by expanding $\exp((y_1+1)\log\frac{1}{1-z})$ in the form of the original generating series of $y_1$, we get

$$\begin{aligned} &\exp((y_1+1)\log\frac{1}{1-z}) \\ &= \sum_{k\geq 0} \mathrm{G}_{y_1^k}^Y(z) y_1^k \\ &= \sum_{k\geq 0} \left(\sum_{N\geq 0} \mathrm{H}_{y_1^k}(N) z^N\right) y_1^k \end{aligned}$$

[9] The examples in this paper are computed with Maple by using our package.

$$= \sum_{N\geq 0}\left(\sum_{k\geq 0}\mathrm{H}_{y_1^k}(N)y_1^k\right)z^N$$

$$= \sum_{N\geq 0}\exp\left(-\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)z^N.$$

Consequently,

$$\begin{aligned}\mathrm{H} &= \exp\left(\mathrm{H}_{y_1}y_1\right)\prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow}\exp(\mathrm{H}_{\Sigma_l}\Pi_l)\\ &= [z^N]\mathrm{G}^Y(z)\\ &\widetilde{{}_{N\to\infty}}\exp\left(-\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\pi_Y Z_{⧢}.\end{aligned}$$

**Example 8.** According to expression (28), we reduce the following relations by identifying local coordinates:

$$\mathrm{H}_{\Sigma_{y_1}}(N) = \ln(N)+\gamma+1/2\,N^{-1}-1/12\,N^{-2}$$
$$+\frac{1}{120}\frac{1}{N^4}+O(N^{-5})$$
$$\mathrm{H}_{\Sigma_{y_2}}(N) = -N^{-1}+1/2\,N^{-2}-1/6\,N^{-3}$$
$$+(\frac{1}{N^4})+\zeta(\Sigma_2)$$
$$\mathrm{H}_{\Sigma_{y_1^2}}(N) = 1/2\,(\ln(N)+\gamma)^2+\frac{1/2\,\ln(N)+1/2\,\gamma}{N}$$
$$+\frac{-1/12\,\ln(N)-1/12\,\gamma+1/8}{N^2}-1/24\,N^{-3}$$
$$+(\frac{1}{120}\,\ln(N)+\frac{1}{120}\,\gamma+\frac{1}{288})\frac{1}{N^4}+O(N^{-5})$$
$$\mathrm{H}_{\Sigma_{y_3}}(N) = -1/2\,N^{-2}+1/2\,N^{-3}-1/4\,\frac{1}{N^4}$$
$$+\zeta(\Sigma_3)+O(N^{-5})$$
$$\mathrm{H}_{\Sigma_{y_2y_1}}(N) = 1/2\,\zeta(\Sigma_3)+\frac{1+\ln(N)+\gamma}{N}$$
$$+\frac{-1/2-1/2\,\gamma-1/2\,\ln(N)}{N^2}$$
$$+(\frac{7}{18}+1/6\,\ln(N)+1/6\,\gamma)N^{-3}-\frac{5}{24}\frac{1}{N^4}$$
$$+O(N^{-5}).$$

### 3.3 Representations of polyzetas

As we see the definition of polyzetas at (2), these convergent series are also compatible with the stuffle product like multiple harmonic sums. Using the expression in Proposition 6, we set

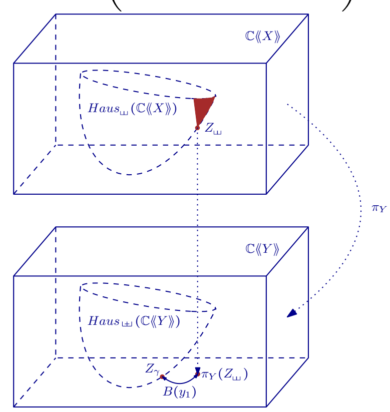
$$Z_{⧺} := \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow}\exp(\zeta(\Sigma_l)\Pi_l). \qquad (28)$$

On the other hand, we conclude from (16) that polyzetas are also obtained by letting $z \longrightarrow 1$ in multiple polylogarithms. Due to the isomorphism in the algebraic structures, we establish a bridge equation between the generating series $\mathbb{Z}_{⧺}$ and $\mathbb{Z}_{⧢}$ as follows.

**Proposition 7** ([9]). *We have a bridge equation between the two spaces $\mathbb{C}\langle\langle X\rangle\rangle$ and $\mathbb{C}\langle\langle Y\rangle\rangle$:*

$$Z_{⧺} = B'(y_1)\pi_Y(Z_{⧢}), \qquad (29)$$

*where* $B'(y_1) = \exp\left(\sum_{k\geq 2}\frac{(-1)^{k-1}\zeta(k)}{k}y_1^k\right)$.




Let $\mathcal{Z}_n$ be the $\mathbb{Q}$-vector space generated by polyzetas of weight $n$. Using this formula, we can rewrite the two sides on the same transcendence basis and then reduce the relations among polyzetas by identifying the local coordinates. On the one hand, by expressing the right hand side of (29) on the basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$, we can identify coefficients on this basis [9, 10]:

**Example 9.** Relations of polyzetas in terms of irreducible elements indexed by the basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$:
Weight 3: $\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$.
Weight 4: $\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$, $\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$, $\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$.
Weight 5:

$$\begin{aligned}\zeta(\Sigma_{y_3y_2}) &= 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-5\zeta(\Sigma_{y_5}),\\ \zeta(\Sigma_{y_4y_1}) &= -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{2}\zeta(\Sigma_{y_5}),\\ \zeta(\Sigma_{y_2^2y_1}) &= \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-\frac{25}{12}\zeta(\Sigma_{y_5}),\\ \zeta(\Sigma_{y_3y_1^2}) &= \frac{5}{12}\zeta(\Sigma_{y_5}),\\ \zeta(\Sigma_{y_2y_1^3}) &= \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{4}\zeta(\Sigma_{y_5}).\end{aligned}$$

Weight 6:

$$\begin{aligned}\zeta(\Sigma_{y_6}) &= \frac{8}{35}\zeta(\Sigma_{y_2})^3,\\ \zeta(\Sigma_{y_4y_2}) &= \zeta(\Sigma_{y_3})^2-\frac{4}{21}\zeta(\Sigma_{y_2})^3,\\ \zeta(\Sigma_{y_5y_1}) &= \frac{2}{7}\zeta(\Sigma_{y_2})^3-\frac{1}{2}\zeta(\Sigma_{y_3})^2,\\ \zeta(\Sigma_{y_3y_1y_2}) &= -\frac{17}{30}\zeta(\Sigma_{y_2})^3+\frac{9}{4}\zeta(\Sigma_{y_3})^2,\\ \zeta(\Sigma_{y_3y_2y_1}) &= 3\zeta(\Sigma_{y_3})^2-\frac{9}{10}\zeta(\Sigma_{y_2})^3,\\ \zeta(\Sigma_{y_4y_1^2}) &= \frac{3}{10}\zeta(\Sigma_{y_2})^3-\frac{3}{4}\zeta(\Sigma_{y_3})^2,\\ \zeta(\Sigma_{y_2^2y_1^2}) &= \frac{11}{63}\zeta(\Sigma_{y_2})^3-\frac{1}{4}\zeta(\Sigma_{y_3})^2,\end{aligned}$$

 pISSN 1859-1388
eISSN 2615-9678

$$\begin{aligned}
\zeta(\Sigma_{y_3y_1^3}) &= \frac{1}{21}\zeta(\Sigma_{y_2})^3,\\
\zeta(\Sigma_{y_2y_1^4}) &= \frac{17}{50}\zeta(\Sigma_{y_2})^3+\frac{3}{16}\zeta(\Sigma_{y_3})^2.
\end{aligned}$$

Weight 7:

$$\begin{aligned}
\zeta(\Sigma_{y_4y_3}) &= \frac{35}{2}\zeta(\Sigma_{y_7})-10\zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5}),\\
\zeta(\Sigma_{y_5y_2}) &= 5\zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5})-\frac{21}{2}\zeta(\Sigma_{y_7})\\
&+ \frac{4}{5}\zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3}),\\
\zeta(\Sigma_{y_6y_1}) &= -\zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5})-\frac{2}{5}\zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})\\
&+ \frac{7}{2}\zeta(\Sigma_{y_7}),\\
\zeta(\Sigma_{y_3y_2^2}) &= \frac{3}{2}\zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})-\frac{217}{48}\zeta(\Sigma_{y_7}),\\
\zeta(\Sigma_{y_3^2y_1}) &= \frac{7}{24}\zeta(\Sigma_{y_7}),\\
\zeta(\Sigma_{y_4y_1y_2}) &= \frac{1}{10}\zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})+\frac{7}{48}\zeta(\Sigma_{y_7}),\\
&\dots
\end{aligned}$$

Weight 8

$$\begin{aligned}
\zeta(\Sigma_{y_8}) &= \frac{24}{175}\zeta(\Sigma_{y_2})^4,\\
\zeta(\Sigma_{y_5y_3}) &= \frac{126}{25}\zeta(\Sigma_{y_2})^4-720\zeta(\Sigma_{y_3y_1^5}),\\
\zeta(\Sigma_{y_6y_2}) &= -\frac{282}{125}\zeta(\Sigma_{y_2})^4+2\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5})\\
&+ 288\zeta(\Sigma_{y_3y_1^5}),\\
\zeta(\Sigma_{y_7y_1}) &= \frac{6}{25}\zeta(\Sigma_{y_2})^4-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5}),\\
\zeta(\Sigma_{y_3^2y_2}) &= \frac{9}{2}\zeta(\Sigma_{y_3})^2\zeta(\Sigma_{y_2})-\frac{44}{5}\zeta(\Sigma_{y_2})^4\\
&- 15\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5})+1440\zeta(\Sigma_{y_3y_1^5}),\\
&\dots
\end{aligned}$$

On the other hand, we use the inverse of $\pi_Y$, denoted by $\pi_X$, to express equality (29) on the basis $\{S_l\}_{l\in\mathcal{L}ynX}$. It means that $\pi_X$ is a morphism defined on the word by $\pi_X(y_{s_1}\dots y_{s_r})=x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1$, and applying to the two sides of (29) we have

$$\pi_X(Z_{\uplus}) = B'(x_1)Z_{\sqcup\!\sqcup}. \tag{30}$$

Hence, we can represent the left hand side of this bridge equation on the basis $\{S_l\}_{l\in\mathcal{L}ynX}$ and then identifying local coordinates to reduce polynomial relations among polyzetas [9, 10].

**Example 10.** Relations of polyzetas in terms of irreducible elements indexed by the basis $\{S_l\}_{l\in\mathcal{L}ynX}$:
Weight 3: $\zeta(S_{x_0x_1^2})=\zeta(S_{x_0^2x_1})$
Weight 4:

$$\begin{aligned}
\zeta(S_{x_0^3x_1}) &= \frac{2}{5}\zeta(S_{x_0x_1})^2,\\
\zeta(S_{x_0^2x_1^2}) &= \frac{1}{10}\zeta(S_{x_0x_1})^2,\\
\zeta(S_{x_0x_1^3}) &= \frac{2}{5}\zeta(S_{x_0x_1})^2.
\end{aligned}$$

Weight 5:

$$\begin{aligned}
\zeta(S_{x_0^3x_1^2}) &= -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1}),\\
\zeta(S_{x_0^2x_1x_0x_1}) &= -\frac{3}{2}\zeta(S_{x_0^4x_1})+\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}),\\
\zeta(S_{x_0^2x_1^3}) &= -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1}),\\
\zeta(S_{x_0x_1x_0x_1^2}) &= \frac{1}{2}\zeta(S_{x_0^4x_1}),\ \zeta(S_{x_0x_1^4})=\zeta(S_{x_0^4x_1}).
\end{aligned}$$

Weight 6:

$$\begin{aligned}
\zeta(S_{x_0^5x_1}) &= \frac{8}{35}\zeta(S_{x_0x_1})^3,\\
\zeta(S_{x_0^4x_1^2}) &= \frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2,\\
\zeta(S_{x_0^3x_1x_0x_1}) &= \frac{4}{105}\zeta(S_{x_0x_1})^3,\\
\zeta(S_{x_0^3x_1^3}) &= \frac{23}{70}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2,\\
\zeta(S_{x_0^2x_1x_0x_1^2}) &= \frac{2}{105}\zeta(S_{x_0x_1})^3,\\
\zeta(S_{x_0^2x_1^2x_0x_1}) &= -\frac{89}{210}\zeta(S_{x_0x_1})^3+\frac{3}{2}\zeta(S_{x_0^2x_1})^2,\\
\zeta(S_{x_0^2x_1^4}) &= \frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2,\\
\zeta(S_{x_0x_1x_0x_1^3}) &= \frac{8}{21}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2,\\
\zeta(S_{x_0x_1^5}) &= \frac{8}{35}\zeta(S_{x_0x_1})^3.
\end{aligned}$$

Weight 7

$$\begin{aligned}
\zeta(S_{x_0^5x_1^2}) &= -\zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1})\\
&- \frac{2}{5}\zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})+3\zeta(S_{x_0^6x_1}),\\
\zeta(S_{x_0^4x_1x_0x_1}) &= -5\zeta(S_{x_0^6x_1})+3\zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1}),\\
\zeta(S_{x_0^4x_1^3}) &= -\frac{23}{60}\zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})\\
&- 2\zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1})+5\zeta(S_{x_0^6x_1}),\\
\zeta(S_{x_0^3x_1x_0^2x_1}) &= 2\zeta(S_{x_0^6x_1})-\zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1}),\\
\zeta(S_{x_0^3x_1x_0x_1^2}) &= -\frac{1}{20}\zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})\\
&- \frac{1}{2}\zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1})+\frac{19}{16}\zeta(S_{x_0^6x_1}),\\
&\dots
\end{aligned}$$

Weight 8

$$\begin{aligned}
\zeta(S_{x_0^7x_1}) &= \frac{24}{175}\zeta(S_{x_0x_1})^4,\\
\zeta(S_{x_0^6x_1^2}) &= \frac{6}{35}\zeta(S_{x_0x_1})^4-\zeta(S_{x_0^4x_1})\zeta(S_{x_0^2x_1}),\\
\zeta(S_{x_0^5x_1x_0x_1}) &= -\frac{6638}{30625}\zeta(S_{x_0x_1})^4\\
&- \frac{149753}{4725}\zeta(S_{x_0^4x_1})\zeta(S_{x_0^2x_1})\\
&- \frac{29026}{1575}\zeta(S_{x_0^2x_1})^2\zeta(S_{x_0x_1})\\
&+ \frac{2}{5}\zeta(S_{x_0x_1^2x_0x_1^4}),
\end{aligned}$$

$$\begin{aligned}\zeta(S_{x_0^5x_1^3}) &= -\frac{11}{4}\zeta(S_{x_0^4x_1})\zeta(S_{x_0^2x_1})\\ &+ \frac{1}{2}\zeta(S_{x_0^2x_1})^2\zeta(S_{x_0x_1})+\frac{61}{175}\zeta(S_{x_0x_1})^4,\\ &\ldots\end{aligned}$$

The above examples, one again, show us that each polyzetas only has a linear representation of plyzetas of the same weight. Hence, we can list the elements of linear bases of $\mathcal{Z}$ corresponding to the bases $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ and $\{S_l\}_{l\in\mathcal{L}ynX}$ that confirm the Zagier's dimension conjecture[10] (see [4]). Moreover, we can reduce algebraic bases (the normal product) from these representations. We show here two lists of irreducible elements up to weight 12 (see more [11]):

1. For the basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$: $\zeta(\Sigma_{y_2}),\zeta(\Sigma_{y_3})$, $\zeta(\Sigma_{y_5})$, $\zeta(\Sigma_{y_7}),\zeta(\Sigma_{y_3y_1^5}),\zeta(\Sigma_{y_9})$, $\zeta(\Sigma_{y_3y_1^7})$, $\zeta(\Sigma_{y_{11}}),\zeta(\Sigma_{y_2y_1^9}),\zeta(\Sigma_{y_2^2y_1^8}),\zeta(\Sigma_{y_3y_1^9})$.
2. For the basis $\{S_l\}_{l\in\mathcal{L}ynX}$: $\zeta(S_{x_0x_1}),\zeta(S_{x_0^2x_1})$, $\zeta(S_{x_0^6x_1}),\zeta(S_{x_0^6x_1})$, $\zeta(S_{x_0x_1^2x_0x_1^4}),\zeta(S_{x_0^8x_1})$, $\zeta(S_{x_0x_1^2x_0x_1^6})$, $\zeta(S_{x_0^{10}x_1})$, $\zeta(S_{x_0x_1^2x_0x_1^2x_0x_1^4})$, $\zeta(S_{x_0x_1x_0x_1^9})$, $\zeta(S_{x_0^3x_1x_0x_1^7})$.

## 4 Conclusion

We represented special functions (multiple harmonic sums, polyzetas, and multiple polylogarithms) in forms of non-commutative generating series indexed by transcendence bases of quasi-shuffle algebras. By identifying the local coordinates of the Hausdorff groups, in shuffle and stuffle Hopf algebras, we can reduce polynomial relations or asymptotic expansions of these special functions indexed by the bases.

## References


1. Chien BV, Duchamp GHE, Minh VHN. Schützenberger's factorization on the (completed) Hopf algebra of q-stuffle product. JP J Algebra Number Theory Appl. 2013;30(2):191-215.
2. Reutenauer C. Free Lie algebras, volume 7 of London Mathematical Society Monographs. Oxford: Oxford University Press; 1993.
3. Minh HN, Petitot M, Hoeven JVD. Shuffle algebra and polylogarithms. Discrete Mathematics. 2000;225(1-3):217-230.
4. Zagier D. Values of Zeta Functions and Their Applications. In: Joseph A, Mignot F, Murat F, Prum B, Rentschler R, editors. First European Congress of Mathematics Paris, July 6–10, 1992: Vol II: Invited Lectures (Part 2). Basel: Birkhäuser Basel; 1994. p. 497-512.
5. Ree R. Lie elements and an algebra associated with shuffles. Ann of Math. 1958;68(2):210-220.
6. Kuo-tsai C. Algebras of iterated path integrals and fundamental groups. Trans Amer Math Soc. 1971 ;156:359-379.
7. Hoffman ME. The Algebra of Multiple Harmonic Series. Journal of Algebra. 1997;194(2):477-495.
8. Radford DE. A natural ring basis for the shuffle algebra and an application to group schemes. Journal of Algebra. 1979;58(2):432-454.
9. Chien BV, Duchamp GHE, Minh VHN. Structure of polyzetas and explicit representation on transcendence bases of shuffle and stuffle algebras. J Symbolic Comput. 2017;83:93-111.
10. Chien BV. Développement asymptotique des sommes harmoniques[dissertation]. Paris: Laboratoire LIPN – Université Paris 13; 2016.
11. Minh HN, Petitot M. Lyndon words, multiple polylogarithms and the Riemannfunction. Discrete Math. 2000;217(1-3):273-292.