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ON THE HILBERT COEFFICIENTS AND REDUCTION NUMBERS

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Abstract. Let (R, m) be a noetherian local ring with $dim(R) = d \geq 1$ and $depth(R) \geq d - 1$. Let I be an m-primary ideal of R. In this paper, we study the non-positivity of the Hilbert coefficients $e_i(I)$ under some conditions.

Keywords: Hilbert coefficients, reduction numbers, Castelnuovo-Mumford regularity, m-primary ideals, the depth of associated graded rings

1 Introduction

Let (R, m) be a noetherian local ring of dimension $d \geq 1$ and I an m-primary ideal of R. Let $\ell(.)$ denote the length of an R-module. The Hilbert-Samuel function of R with respect to I is the function $H_I: \mathbb{Z} \longrightarrow \mathbb{N}_0$ given by

$$H_I(n) = \begin{cases} \ell(R/I^n) & if\ n\ \geq\ 0; \\ 0 & if\ n\ < 0. \end{cases}$$

There exists a unique polynomial $P_I(x) \in \mathbb{Q}[x]$ (called the *Hilbert- Samuel polynomial*) of degree d such that $H_I(n) = P_I(n)$ for $n \gg 0$ and it is written by

$$P_I(n) = \sum_{i=0}^{d} (-1)^i \binom{n + d - i - 1}{d - i} e_i(I).$$

Then, the integers $e_i(I)$ is called *Hilbert coefficients* of I. The aim of this paper is to study the non-positivity of Hilbert cofficients.

In 2010, Mandal-Singh-Verma [1] showed that $e_1(I) \leq 0$ for all parameter ideals I of R. If $depth(R) \geq d - 1$, McCune [2] showed that $e_2(I) \leq 0$ and Saikia-Saloni [3] proved that $e_3(I) \leq 0$ for every parameter ideal I. Recently, Linh-Trung [4] proved that if $depth(R) \geq d - 1$ and I is a parameter ideal such that $depth(G(I)) \geq d - 2$, then $e_i(I) \leq 0$ for $i = 1, \ldots, d$. In [5], Puthenpurakal obtained remarkable results that if I is an m-primary ideal of a ring R with dimension 3 such that $r(I) \leq 2$, then $e_3(I) \leq 0$.

The main result of this paper is to give an improvement of the result of Linh-Trung [4].

Theorem 3.3 *Let (R, m) a noetherian local ring with $dim(R) = d \geq 2$ and $depth(R) \geq d - 1$. Let I be an m-primary ideal of R such that $depth(G(I)) \geq d - 2$. For $i = 1, \ldots, d$, if $r(I) \leq i - 1$ then $e_i(I) \leq 0$.*

2 Preliminary

Let (R, m) be a noetherian local ring of dimension d and I be an m-primary ideal of R. A numerical function

$H_I: \mathbb{Z} \longrightarrow \mathbb{N}_0$

$$n \longmapsto H_I(n) = \begin{cases} \ell(R/I^n) & if\ n\ \geq\ 0; \\ 0 & if\ n\ < 0. \end{cases}$$

is said to be a *Hilbert-Samuel function* of R with respect to the ideal I. It is well known that there exists a polynomial $P_I \in \mathbb{Q}[x]$ of degree d such that $H_I(n) = P_I(n)$ for $n \gg 0$. The polynomial P_I is called the *Hilbert-Samuel polynomial* of R with respect to the ideal I, and it is written in the form

$$P_I(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(I),$$

where $e_i(I)$ for $i = 0, \dots, d$ are integers, called *Hilbert coefficients* of I. In particular, $e(I) = e_0(I)$ and $e_1(I)$ are called the *multiplicity* and *Chern coefficient* of I, respectively.

An element $x \in I \backslash mI$ is said to be *superficial* for I if there exists a number $c \in \mathbb{N}$ such that $(I^n : x) \cap I^c = I^{n-1}$ for $n > c$. If R/m is infinite, then a superficial element for I always exists. A sequence of elements $x_1, \dots, x_r \in I \backslash mI$ is said to be a *superficial sequence* for I if x_i is superficial for $I/(x_1, \dots, x_{i-1})$ for $i = 1, \dots, r$.

Suppose that $dim(R) = d \geq 1$ and $x \in I \backslash mI$ is a superficial element for I, then $\ell(0:_R x) < \infty$ and $dim(R/(x)) = dim(R) - 1 = d - 1$. The following lemma give us a relationship between $e_i(I)$ and $e_i(I_1)$, where $I_1 = I(R/(x))$.

Lemma 2.1 *[6, Proposition 1.3.2] Let R be a noetherian local ring of dimension $d \geq 2$ and I an m-primary ideal of R. Let $x \in I \backslash mI$ be a superficial element for I and $I_1 = I(R/(x))$. Then*

(i) $e_i(I) = e_i(I_1)$ for $i = 0, \dots, d-2$;

(ii) $e_{d-1}(I) = e_{d-1}(I_1) + (-1)^d \ell(0:x)$.

If denote by $G(I) = \oplus_{n \geq 0} I^n / I^{n+1}$ the associated graded ring of R with respect to I and

$$a_i(G(I)) = \sup\{n \mid H^i_{G(I)_+}(G(I))_n \neq 0\},$$

where $H^i_{G(I)_+}(G(I))$ is the i-th local cohomology module of $G(I)$ with respect to $G(I)_+$. The *Castelnuovo-Mumford regularity* of $G(I)$, $reg(G(I))$, is defined by

$$reg(G(I)) = max\{a_i(G(I)) + i \mid i \geq 0\}.$$

Recall that an ideal $J \subseteq I$ is called a *reduction* of I if $I^{n+1} = JI^n$ for $n \gg 0$. If J is a reduction of I and no other reduction of I is contained in J, then J is said to be a *minimal reduction* of I. If J is a minimal reduction of I, then the *reduction number of I with respect to J*, $r_J(I)$, is given by

$$r_J(I) := \min\{ n \mid I^{n+1} = JI^n\}.$$

The *reduction number* of I, denoted $r(I)$, is given by

$$r(I) := \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

A relationship between the reduction number of I, $a_d(G(I))$ and $reg(G(I))$ is given by the following lemma.

Lemma 2.2 *[7, Proposition 3.2]*

$$a_d(G(I)) + d \leq r(I) \leq reg(G(I)).$$

3 Main result

Throughout this section, (R, m) is a noetherian local ring of dimension d and $depth(R) \geq d - 1$. Let I be an m-primary ideal of R. In [8], Elias considered the numerical function

$$\begin{aligned} \sigma_I : \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto \sigma_I(k) = depth(G(I^k)). \end{aligned}$$

Elias [8] showed that σ_I is a non-decreasing function and $\sigma_I(k)$ is a constant for $k \gg 0$. This constant is denoted by $\sigma(I)$.

By [9, Lemma 2.4],

$$a_i(G(I^k)) \leq \left[\frac{a_i(G(I))}{k}\right] \text{ for all } i \leq d \text{ and } k \geq 1,$$

where $[a] = max\{m \in \mathbb{Z} \mid m \leq a\}$. Thus, for $i \geq 0$, we have

$$a_i(G(I^k)) \leq 0 \quad \text{for} \quad k \gg 0 \qquad (1)$$

and

$$\sigma(I) \geq depth(G(I)). \qquad (2)$$

The following theorem gives a non-positivity for the last Hilbert coefficient.

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Theorem 3.1 *[10, Theorem 2.4] Let (R,m) be a noetherian local ring of dimension $d \geq 2$ and $depth(R) \geq d-1$. Let I be an m-primary ideal such that $r(I) \leq d-1$ and $\sigma(I) \geq d-2$. Then, $e_d(I) \leq 0$.*

For $k \gg 0$, let $J = I^k$, $R = R[Jt] = \oplus_{n\geq 0} J^n$ denote the Rees algebra of R with respect to J, $R_+ = \oplus_{n>0} R_n$. By [11, Theorem 4.1] and [11, Theorem 3.8], we have

$$(-1)^d e_d(I) = (-1)^d e_d(J) = P_J(0) - H_J(0)$$
$$= \sum_{i=0}^d (-1)^i \ell(H^i_{R_+}(R)_0)$$
$$= \sum_{i=0}^d (-1)^i \ell(H^i_{G(J)_+} G(J)_0).$$

Since $\sigma(I) = depth(G(J)) \geq d-2$, $H^i_{G(I)_+}(G(I)) = 0$ for $i = 0, \dots, d-3$. By Lemma 2.2, we have $a_d(G(J)) + d \leq r(J)$. From [9, Lemma 2.7],

$$r(J) \leq \frac{]r(I)+1-s(I)[}{k} + s(I) - 1$$
$$= \frac{]r(I)+1-d[}{k} + d - 1 \leq d-1.$$

Hence, $a_d(G(J)) < 0$. On the other hand, $a_i(G(J)) \leq 0$ for all $i \geq 0$ from (1). By applying [12, Theorem 5.2], we get $a_{d-2}(G(J)) < a_{d-1}(G(J)) \leq 0$. It follows that

$$(-1)^d e_d(I) = (-1)^{d-1} \ell(H^{d-1}_{G(J)_+} G(J)_0).$$

This implies that $e_d(I) = -\ell(H^{d-1}_{G(J)_+}(G(J))_0) \leq 0$. From the proof of Theorem 3.1, we obtain the following corollary.

Corollary 3.2 *Let (R,m) be a noetherian local ring of dimension $d \geq 2$ and $depth(R) \geq d-1$. Let I be an m-primary ideal such that $reg(G(I)) \leq d-2$ and $\sigma(I) \geq d-2$. Then, $e_d(I) = 0$.*

For $k \gg 0$, set $J = I^k$. Since $reg(G(I)) \leq d-2$,

$$max\{a_{d-1}(G(I)) + d - 1, a_d(G(I)) + d\} \leq d-2.$$

Thus, $a_i(G(I)) \leq -1$ for $i = d-1, d$. By [9, Lemma 2.4],

$$a_i(G(I^k)) \leq [a_i(G(I))/k].$$

Therefore,

$$max\{a_{d-1}(G(J)), a_d(G(J))\} \leq -1.$$

From the proof of Theorem 3.1, we have

$$e_d(I) = -\ell(H^{d-1}_{G(J)_+}(G(J))_0) = 0.$$

In [4], Linh-Trung proved that if Q is a parameter ideal such that $depth(G(Q)) \geq d-2$, then $e_i(Q) \leq 0$ for all $i = 1, \dots, d$. In this case, $r(Q) = 0$. The following theorem is an improvement of Linh-Trung's result.

Theorem 3.3 *Let (R,m) a noetherian local ring with $dim(R) = d \geq 2$ and $depth(R) \geq d-1$. Let I be an m-primary ideal of R such that $depth(G(I)) \geq d-2$. For $i = 1, \dots, d$, if $r(I) \leq i-1$, then $e_i(I) \leq 0$.*

It is clear that the theorem holds for $d = 2$.

Now, we consider $d > 2$. By [4, Theorem 1], the theorem holds for the case $i = 1$.

In the case $i = d$, by assumption, we have $r(I) \leq d-1$ and $\sigma(I) \geq depth(G(I)) \geq d-2$. By applying Theorem 3.1, we obtain $e_d(I) \leq 0$. So, we need to prove for $i = 2, \dots, d-1$.

Without loss of generality, we assume that R/m is infinite and $x_1, \dots, x_{d-i}$ is a superficial sequence for I. Let $R_i = R/(x_1, \dots, x_i)$ and $I_i = IR_i$. Then, $e_i(I) = e_i(I_{d-i})$ from Lemma 2.1. From this hypothesis, it follows that

$$dim(R_{d-i}) = i \geq 2, \quad depth(R_{d-i}) \geq i-1 \quad \text{and } depth(G(I_{d-i})) \geq i-2.$$

We have $r(I_{d-i}) \leq r(I) \leq i-1$. From (2), we get $\sigma(I_{d-i}) \geq depth(G(I_{d-i})) \geq i-2$. Appying Theorem 3.1, we obtain

$$e_i(I) = e_i(I_{d-i}) \leq 0 \quad \text{for} \quad i = 2, \dots, d-1.$$

The proof is complete.

Combining Theorem 3.3 and Corollary 3.2, we get the following corollary.

Corollary 3.4 *Let (R,m) be a noetherian local ring with $dim(R) = d \geq 2$ and $depth(R) \geq d-1$.*

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Let I be an $\mathfrak{m}$-primary ideal of R such that $depth(G(I)) \geq d-2$. For $i = 1, \ldots, d$, if $reg(G(I)) \leq i-2$, then $e_i(I) = 0$.

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