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The local uniform convergence of positive harmonic function sequence

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Abstract. The Harnack distance on space $\mathbb{R}^n$ and its conformal invariance were constructed and studied by Herron. In this paper, we obtain the Harnack distance on domains D in $\mathbb{C}$. Then, we use this concept to investigate some properties of the positive harmonic function class. These results are obtained in the complex plane, so it is advantageous to take some tools of the complex analysis. The main result of this paper is the property of the local uniform convergence of the positive harmonic sequences on a domain in the complex plane.

Keywords: harmonic function, Harnack distance, local uniform convergence

1 Introduction

Harmonic functions and subharmonic functions are the main research subject of potential theory. These function classes are usually presented in space $\mathbb{R}^n$ [1-3]. In [4], the author constructed the Harnack distance on space $\mathbb{R}^n$ and studied its conformal invariance. This is convenient to use the differential operators and integral operators that are common in the theory of multivariable functions. However, it does not take advantage of the powerful tools of complex analysis. Moreover, it is hard to expand many results to the pluripotential theory [3]. In this paper, we investigate some results of the positive harmonic function class in $\mathbb{C}$. Then, we use these results to prove the property of the local uniform convergence of the positive harmonic sequences.

From this research, we also pose an open question of whether it is possible to extend this result to the new function classes [5-7].

2 Preliminaries

We denote $\mathbb{C}$ be a complex plane. A subset $D \subset \mathbb{C}$ is called a domain if it is open, connected, and non-empty. A disc centred at w with radius δ is denoted and defined as

$$\Delta(w,\delta) := \{z \in \mathbb{C}: |z-w| < \delta\}.$$

Let U be an open subset in $\mathbb{C}$. The function $h: U \to \mathbb{R}$ is harmonic if it is in $C^2(U)$ and satisfies the Laplace equation as follows:

$$\Delta h(z) := \frac{\partial^2 h}{\partial x^2}(z) + \frac{\partial^2 h}{\partial y^2}(z) = 0,$$

for all $z = x + iy \in U$. A positive harmonic function h on U is a harmonic function and $h \geq 0$ on U.

We need the identity principle for the harmonic class to prove the main result. We cite here for convenient reading.

Theorem 2.1 Let h and k be harmonic functions on domain D in $\mathbb{C}$. If $h = k$ on non-empty open subset U of D; then $h = k$ on the whole D.

Proof: If h, k are harmonic functions on D, then $h - k$ is also a harmonic function on D. So, we can suppose, without loss of generality, that $k = 0$. Set $g = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}$. Then, from the proof of Theorem 1 (page 111 [8]), g is holomorphic on D, and also $g = 0$ on U since $h = 0$. From the identity principle for holomorphic functions, it follows that $g = 0$ throughout D, and, hence, that $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ on D. Therefore, h is constant on D; and since $h = 0$ on U, this implies that the constant must be zero. □

One of the most important results for the positive harmonic function class is the Harnack's inequality

Theorem 2.2 Let h be a positive harmonic function on disc $\Delta(w, \delta)$. Then, for all $r < \delta$ and $0 \le t < 2\pi$,

$$\frac{\delta - r}{\delta + r}h(w) \le h(w + re^{it}) \le \frac{\delta + r}{\delta - r}h(w).$$

Proof: see Theorem 2.14 [4]. □

Prompted by the Harnack's inequality, the Harnack distance is defined as follow:

Definition 2.3 Let D be a domain in $\mathbb{C}$. Given $z, w \in D$, the Harnack distance between z and w, denoted by $\tau_D(z, w)$, and defined as

$$\tau_D(z, \omega) = \min\{\tau \in \mathbb{R}: \tau^{-1}h(\omega) \le h(z) \le \tau h(\omega)\},$$

where min is taken over all the positive harmonic functions h on D.

The following result is a formulation of the Harnack distance from a point in a disc to its centre.

Theorem 2.4 If $\Delta = \Delta(w, \delta)$, then

$$\tau_\Delta(z, w) = \frac{\delta + |z - w|}{\delta - |z - w|} \quad (z \in \Delta).$$

Proof: Let $z \in \Delta$. From Harnack's inequality (Theorem 2.2), we have

$$\frac{\rho - |z - \omega|}{\rho + |z - \omega|}h(\omega) \le h(z) \le \frac{\rho + |z - \omega|}{\rho - |z - \omega|}h(\omega)$$

for all positive harmonic functions h on Δ. From the definition of the Harnack distance, we infer

$$\tau_\Delta(z, \omega) \le \frac{\rho + |z - \omega|}{\rho - |z - \omega|}. \quad (1)$$

On the other hand, for each $|\xi| = 1$, let h_ξ be a function definite on Δ as follows:

$$h_\xi(z) = P\left(\frac{z - \omega}{\rho}, \xi\right) = Re\left(\frac{\rho\xi + (z - \omega)}{\rho\xi - (z - \omega)}\right),$$

where P is the Poisson kernel [3]. From the property of the Poisson kernel (Lemma 2.2.1 [3]), we infer that h_ξ is a positive harmonic function on Δ. We have $h_\xi(\omega) = 1$. Set $z = \omega + re^{it}$, where $r = |z - \omega|$ and $\xi = e^{i\theta}$, then

$$h_\xi(z) = Re\left(\frac{\rho e^{i\theta} + re^{it}}{\rho e^{i\theta} - re^{it}}\right) = \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(t - \theta) + r^2}.$$

From the definition of $\tau_\Delta(z, \omega)$, we have

$$\tau_\Delta^{-1}(z, \omega)h_\xi(\omega) \le h_\xi(z) \le \tau_\Delta(z, \omega)h_\xi(\omega).$$

This is equivalent to

$$\tau_\Delta^{-1}(z, \omega)h_\xi(\omega) \le \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(t - \theta) + r^2} \le \tau_\Delta(z, \omega)h_\xi(\omega),$$

for all $0 \le \theta < 2\pi$. Choose $\theta = t$, we have

$$\tau_\Delta^{-1}(z, \omega)h_\xi(\omega) \le \frac{\rho + r}{\rho - r} \le \tau_\Delta(z, \omega)h_\xi(\omega).$$

Sine $h_\xi(\omega) = 1$, so

$$\tau_\Delta(z, \omega) \ge \frac{\rho + r}{\rho - r} = \frac{\rho + |z - \omega|}{\rho - |z - \omega|}. \quad (2)$$

From (1) and (2), we infer that the item is proved. □

In the main result of this paper, we use the Harnack distance to prove a result of the local uniform convergence of the positive harmonic sequence. Let (h_n) be a sequence of functions that are defined on open set U. The sequence (h_n) is said to be point-wise convergence to function h if for each $z \in U$ and $\varepsilon > 0$, there exists $N = N(z, \varepsilon) > 0$ such that

$$|h_n(z) - h(z)| < \varepsilon \quad (\forall\, n \ge N).$$

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Let $E \subset U$. The sequence (h_n) is said to be a uniform convergence to h on E if for each $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, such that

$$|h_n(z) - h(z)| < \varepsilon \quad (\forall z \in E, \forall\, n \geq N).$$

The sequence (h_n) is said to be a local uniform convergence to h on U if for each $z \in U$, there exists a neighborhood E_x of x such that $h_n \to h$ uniformly on E_x.

We also need the following for the proof of the main theorem.

Theorem 2.5 Let (h_n) be positive harmonic functions on domain D in $\mathbb{C}$. Then, either $h_n \to +\infty$ locally uniform, or else some subsequence $h_{n_j} \to h$ locally uniform, where h is harmonic on D.

Proof: Fix $w \in D$. For all $z \in D$ and $n \geq 1$, from the inequalities

$$\tau_D^{-1}(z, w) h_n(w) \leq h_n(z) \leq \tau_D(z, w) h_n(w), \qquad (*)$$

it follows that if $h_n(w) \to +\infty$, then, also $h_n \to +\infty$ locally uniform on D, and if $h_n(w) \to 0$, then, also $h_n \to 0$ locally uniform on D. Therefore, replacing (h_n) by a subsequence if necessary, we can reduce to the case where the sequence $(\log h_n(w))_{n\geq 1}$ is bounded. The inequality $(*)$ then implies that $(\log h_n)_{n\geq 1}$ is locally uniformly bounded on D, and so it suffices to prove that there is a subsequence (h_{n_j}) such that $(\log h_{n_j})_{j\geq 1}$ is locally uniformly convergent on D.

Let S be a countable dense subset of D. The sequence $(\log h_n(\xi))_{n\geq 1}$ is bounded for each $\xi \in S$, so with a "diagonal argument", we may find a subsequence (h_{n_j}) such that $(\log h_{n_j}(\xi))_{j\geq 1}$ is convergent for each $\xi \in S$. We shall show that, for this subsequence, $(\log h_{n_j})_{j\geq 1}$ is locally uninformly convergent on D.

Let K be a compact subset of D, and let $\varepsilon > 0$. For each $z \in K$, let

$$V_z = \{z' \in D : \log \tau_D(z, z') < \varepsilon\},$$

And let $V_{z_1}, \ldots, V_{z_m}$ be a finite subcover of K. As S is dense in D, for each l, we can pick a point $\xi_l \in V_{z_l} \cap S$. Then, there exists $N \geq 1$ such that

$$\left|\log h_{n_j}(\xi_l) - \log h_{n_k}(\xi_l)\right| \leq \varepsilon,$$

for all $n_j, n_k \geq N, l = 1,2,..,m$. According to the definition of the Harnack distance,

$$\left|\log h_{n_j}(z) - \log h_{n_k}(\xi_l)\right| \leq \log \tau_D(z, \xi_l) < 2\varepsilon,$$

for all $z \in V_{z_l}$. Then, we have a similar inequality for h_{n_k}. Hence,

$$\left|\log h_{n_j}(z) - \log h_{n_k}(z)\right| \leq 5\varepsilon \left(n_j, n_k \geq N, z \in K\right).$$

Thus, $(\log h_{n_j})_{j\geq 1}$ is a uniform Cauchy on K, and so uniformly convergent there. □

3 Main results

Theorem 3.1 Let $(h_n)_{n\geq 1}$ be positive harmonic functions on domain $D \subset \mathbb{C}$. If $(h_n)_{n\geq 1}$ converge point-wise for each z in a non-empty open subset of D, then $(h_n)_{n\geq 1}$ converge locally uniformly on the whole of D.

Proof: Following the hypothesis, there exists $\emptyset \neq \omega \subset D$, and ω is open such that the sequence $(h_n)_{n\geq 1}$ converges for every $z \in \omega$. Then, we have

$$h_n \nrightarrow +\infty,$$

on D. According to Theorem 2.5, there exists a subsequence (h_{n_j}) such that

$$h_{n_j} \to h, j \to +\infty$$

is local uniform on D and h is a harmonic function on D.

First, we shall prove that the sequence (h_n) converges point-wise to h on D: we suppose that (h_n) does not converge point-wise to h on D. Then, there exists a point $z_0 \in D$, a number $\varepsilon > 0$ and a subsequence (h_{m_j}) such that

$$\left|h_{m_j}(z_0) - h(z_0)\right| > \varepsilon, (\forall j = 1,2, \ldots).$$

By applying Theorem 2.5 for the subsequence (h_{m_j}), we have two cases that can occur as follows:

- Either the sequence $h_{m_j} \to +\infty$ when $j \to +\infty$ is locally uniform on D. Then, $h = +\infty$ on open subset ω of D. So, according to the identity principle (Theorem 2.1), $h = +\infty$ on D. This is contrary to the hypothesis.

- Or, there exists a subsequence $(h_{m_{jk}})$ such that

$$h_{m_{jk}} \to \hat{h}, k \to +\infty,$$

Is locally uniform on D. Then, $h = \hat{h}$ on open subset ω of D. Again, according to the identity principle, $h = \hat{h}$ on D. According to this and the contrary hypothesis,

$$\left|h_{m_{jk}}(z_0) - \hat{h}(z_0)\right| > \varepsilon, (\forall j = 1,2,\dots).$$

This is contrary to the convergence of the sequence $(h_{m_{jk}})$ of the function $\hat{h}$ on D.

Next, we prove the sequence (h_n) converges locally uniformly to h on D: Let $z_1 \in D$. Choose $\delta > 0$ such that $\Delta(z_1, \delta) \subset D$. We will prove (h_n) converges locally uniformly to h on $\Delta(z_1, \frac{\delta}{2})$. Indeed, given $\varepsilon > 0$. Then, for every $z, w \in \Delta(z_1, \frac{\delta}{2})$ and every positive harmonic function h on D, we have

$$\tau^{-1}_{\Delta(w,\delta)}(z,w)h(w) \le h(z) \le \tau_{\Delta(w,\delta)}(z,w)h(w) (**)$$

where

$$\tau_{\Delta(w,\delta)}(z,w) = \frac{\delta + |z-w|}{\delta - |z-w|} < \frac{\delta + \frac{\delta}{2}}{\delta - \frac{\delta}{2}} = 3.$$

– Applying $(**)$ for $w = z_1$, we infer each point $z \in \Delta(z_1, \frac{\delta}{2})$, there exists a number $M_z > 0$ such that

$$|h(z)| < M_z, \quad (1)$$

for all positive harmonic function h on D.

– Since

$$\lim_{|z-w|\to 0} \frac{\delta + |z-w|}{\delta - |z-w|} = 1,$$

So, there exists $\delta_1 > 0$ such that for every $|z-w| < \delta_1$, we have

$$\frac{\delta + |z-w|}{\delta - |z-w|} - 1 < \frac{\varepsilon}{4M_z}.$$

equivalent to

$$\tau_{\Delta(w,\delta)}(z,w) - 1 < \frac{\varepsilon}{4M_z}. \quad (2)$$

We consider δ_1 – a finite net $\{z_1, z_2, \dots, z_m\} \subset \Delta(z_1, \frac{\delta}{2})$ such that

$$\bigcup_{j=1}^{m} \Delta(z_j, \delta_1) \supset \Delta\left(z_1, \frac{\delta}{2}\right).$$

– Since the sequence (h_n) converges at points $z_1, z_2, \dots, z_m$ to h, so there exists $n_0 \ge 1$ large enough such that for every $n > n_0$, we have

$$\left|h_n(z_j) - h(z_j)\right| < \frac{\varepsilon}{2} \quad (\forall j = 1,2,\dots).$$

Each $w \in \Delta(z_1, \frac{\delta}{2})$ is arbitrary. Then, there exists z_j such that $|w - z_j| < \delta_1$. We have

$$\begin{aligned}|h_n(w) - h(w)| \le & \left|h_n(w) - h_n(z_j)\right| \\ & + \left|h_n(z_j) - h(z_j)\right| \\ & + \left|h(z_j) - h(w)\right|. \quad (3)\end{aligned}$$

For all positive harmonic functions u, we have

$$\begin{aligned}\left(\tau^{-1}_{\Delta(w,\delta)}(z_j,w) - 1\right)u(z_j) & \le u(w) - u(z_j) \\ & \le \left(\tau_{\Delta(w,\delta)}(z_j,w) - 1\right)u(z_j).\end{aligned}$$

From this, we infer

$$\left|u(w) - u(z_j)\right| \le \left(\tau_{\Delta(w,\delta)}(z_j,w) - 1\right)\left|u(z_j)\right|.$$

From (1) and (2), we imply that

$$\left|u(w) - u(z_j)\right| < \frac{\varepsilon}{4M_{z_j}} M_{z_j} = \frac{\varepsilon}{4}. \quad (4)$$

Applying (4) for functions h_n and h, then input them to (3), we imply that

$$|h_n(w) - h(w)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon,$$

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for all $n > n_0$ and all $w \in \Delta(z_1, \frac{\delta}{2})$. □

4 Conclusions

In this note, we prove a property of the local uniform convergence of the positive harmonic sequence with the main result, and it is Theorem 3.1. All the results in this paper are presented in space $\mathbb{R}^2$ equivalent to complex plane $\mathbb{C}$. This allows us to use the powerful tools of complex analysis to demonstrate and prove the results. By the way, we propose an open question of whether this property is valid for new function classes.

References

1. Axler S, Bourdon P, Ramey W. Harmonic function theory. New York: Springer – Verlag; 2001. 264 p.
2. Klimek M. Pluripotential Theory. Oxford: Clarendon Press; 1991.
3. Helms LL. Introduction to potential theory. New York: John Wiley and Sons; 1969.
4. Herron DA. The harnack and other conformally invariant metrics. Kodai Mathematical Journal. 1987;10(1):9-19.
5. Hiep PH. Singularities of plurisubharmonic functions. Pub Hou Sci and Tec; 2016.
6. Quy HN. The topology on the space $\delta\mathcal{E}_\chi$. Univ Iagel Acta Math. 2014;51:61-73.
7. Hung VV, Quy HN. The m-Hessian Operator on Some Weighted Energy Classes of Delta m-Subharmonic Functions. Results in Mathematics. 2020;75(3):112.
8. Khue NV, Hai LM. Functions of One complex variable. Hanoi: Hanoi Vietnam National University Press; 1997.