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Hilbert coefficients of ideals under perturbation of an ideal

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Abstract. Let $(R, \mathfrak{m})$ be a noetherian local ring, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$ in R. In this paper, we will prove the preserve of Hilbert coefficients of R/I with respect to J under J-adic perturbations of I, provided that J is a parameter ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$.

Keywords: Hilbert coefficient, small perturbation, perturbation, Hilbert perturbation index, Cohen-Macaulay deviation, Buchsbaum invariant, extended degree, filter regular sequence, d-sequence

1 Introduction

Let $(R, \mathfrak{m})$ be a noetherian local ring and J an $\mathfrak{m}$-primary ideal of R. For an ideal $I = (f_1, ..., f_r)$ of R, the ideal $I' = (f'_1, ..., f'_r)$ of R is said to be *J-adic perturbation of I* if $f'_i \equiv f_i \mod J^n, i = 1, ..., r$ for $n \gg 0$. If I' is an $\mathfrak{m}$-adic perturbation then I' is called *small perturbation* of I.

The aim of this paper is to study the preserve of Hilbert coefficients of R/I with respect to J under J-adic perturbations of I. More precisely, we wish to estimate the least number N such that Hilbert coefficients of R/I and R/I' with respect to J are the same; that is, $e_j(J, R/I) = e_j(J, R/I')$ for all $n \geq 0$, $j = 0, ..., \dim R/I$ and $f'_i \equiv f_i \mod J^N, i = 1, ..., r$. Such the number N is called the *Hilbert perturbation index* of I with respect to J, denoted by $N(J, I)$.

The preserve of the Hilbert function under perturbation has been studied by many mathematicians. Srinivas and Trivedi in 1996 [1] showed that the Hilbert-Samuel function of R/I is preserved under small perturbations of I if R is a generalized Cohen-Macaulay and I is generated by a part of system of parameters. Srinivas and Trivedi also conjectured that the same is true if R is an arbitrary local ring and $f_1, ..., f_r$ is a filter-regular sequence. This conjecture was solved by Ma, Quy and Smirnov [2] when they proved the stronger result that for J-adic perturbation and $J + I$ is an $\mathfrak{m}$-primary ideal of R. Later, Quy and N. V. Trung [3] proved that $G_J(R/I) \cong G_J(R/I')$ under J-adic perturbation for arbitrary ideal J, which implies that the conjecture of Srinivas and Trivedi is true and they gave an explicit upper bound for the Hilbert perturbation index N of I with respect J. Quy and N. V. Trung also proved that $G_J(R/I) \cong G_J(R/I')$ under J-adic perturbation if and only if I is generated by a J-filter regular sequence.

Our main result is to give a linear bound for $N(J, I)$ in terms of the Cohen-Macaulay deviation provided that R is a noetherian local ring and I is an ideal generated by a filter regular sequence of R and J is a parameter ideal generated by a d-sequence of $R/(f_1,...,f_i)$ for $i = 1,...,r$.

Theorem 3.6 *Let $(R,\mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1,...,f_r)$ an ideal generated by a filter regular sequence $f_1,...,f_r$. Suppose that J is an ideal generated by a d-sequence of $R/(f_1,...,f_i)$ for $i = 1,...,r$. Set $s = d - r$ and $\Delta_i = \Delta(J, R/(f_1,...,f_{i-1})), i = 1,...,r$. Let $\Delta = \max\{\Delta_i \mid i = 1,...,r\}$ and*

$$N = (2^r - 1)\Delta + 1.$$

Then for every $\varepsilon_1, \ldots, \varepsilon_r \in J^N$,

$$e_j(J, R/I) = e_j(J, R/I')$$

for $0 \le j \le s$ and $I' = (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r)$.

2 Preliminary

Let $(R,\mathfrak{m})$ be a noetherian local ring and J an $\mathfrak{m}$-primary ideal of R. For each finitely generated R-module M of dimension d, denote $\lambda(.)$ the length of finitely generated R-module, the numerical function

$$H_M : \mathbb{Z} \longrightarrow \mathbb{N}_0$$
$$n \longmapsto H_M(n) = \begin{cases} \lambda(M/J^nM) & \text{if } n \ge 0, \\ 0 & \text{if } n < 0; \end{cases}$$

is said to be a *Hilbert-Samuel function* of M with respect to J. It is well known that there exists a polynomial $P_M \in \mathbb{Q}[x]$ of degree d such that $H_M(n) = P_M(n)$ for $n \gg 0$. The polynomial P_M is called the *Hilbert-Samuel polynomial* of M with respect to J and it is written in the form

$$P_M(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(J, M),$$

where $e_i(J,M)$ for $i = 0,...,d$ are integers, called *Hilbert coefficients* of M with respect to J. In particular, $e(J,M) := e_0(J,M)$ is called the *multiplicity* of M with respect to J.

We write $Spec(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal of } R\}$. Let M be a finitely generated R-module, an ideal $\mathfrak{p} \in Spec(R)$ is said to be *associated to* M if $\mathfrak{p} = (0 :_R m)$ for some $m \in M$. We denote by $\mathrm{Ass}_R(M)$ the set of associated primes of M. A system of elements $\{f_1,...,f_r\} \subseteq \mathfrak{m}$ is called a *filter regular sequence* of M if

$$f_i \in \mathfrak{p} \text{ for all } \mathfrak{p} \in (\mathrm{Ass}_R(M/(f_1,...,f_{i-1})M)) \setminus \{\mathfrak{m}\}$$

for all $i = 1,...,r$.

For each $j \in \mathbb{Z}$, we set

$$M_j = \mathrm{Hom}_R(H^j_{\mathfrak{m}}(M), E),$$

where $E = E_R(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$. Then M_j is a finitely generated R-module with $\dim_R M_j \le j$. For each finitely generated R-module M with $d = \dim M$ and for each $\mathfrak{m}$-primary ideal J of R, the *homological degree* $\mathrm{hdeg}(J,M)$ of M with respect to J is defined by

$$\mathrm{hdeg}(J,M) = \begin{cases} \lambda(M) & \text{if } d \le 0; \\ e(J,M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(J, M_i) & \text{if } d > 0. \end{cases}$$

The homological degree of a module M with respect to an $\mathfrak{m}$-primary ideal J was introduced in [4], a generalization of the notion of homological degree $\mathrm{hdeg}(M)$ defined by Vasconcelos [5]; that is,

$$\mathrm{hdeg}(\mathfrak{m}, M) = \mathrm{hdeg}(M).$$

An another way, if R is a homomorphic image of Gorenstein S of dimension n

$$\mathrm{hdeg}(J,M) = \begin{cases} \lambda(M) & \text{if } d \le 0; \\ e(J,M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(J, \mathrm{Ext}_S^{n-i}(M,S)) & \text{if } d > 0. \end{cases}$$

We can verify that $\mathrm{hdeg}(J,M)$ is an extended degree of M with respect to J (see [5] for the case $J = \mathfrak{m}$).

We denote by

$$\Delta(J,M) = \mathrm{hdeg}(J,M) - e(J,M),$$

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is called the *Cohen-Macaulay deviation* of M with respect to J. In the case $M = R$, we use $\Delta(J) := \Delta(J, R)$. If $J = \mathfrak{m}$, we write $\Delta(M) = \Delta(\mathfrak{m}, M)$.

Recall that a sequence $x_1, ..., x_s$ of R is said to be *d-sequence* if it satisfies one of the following three equivalent conditions:

(i) $(x_1, ..., x_{i-1}) : x_i x_k = (x_1, ..., x_{i-1}) : x_k$ for $1 \leq i \leq k \leq s$;

(ii) $(x_1, ..., x_{i-1}) : x_i) \cap \mathfrak{q} = (x_1, ..., x_{i-1})$ for $1 \leq i \leq s$ and $\mathfrak{q} = (x_1, ..., x_s)$;

(iii) $(x_1, ..., x_{i-1}) : x_i = \bigcup_{n=1}^{\infty} (x_1, ..., x_{i-1}) : \mathfrak{q}^n$ and $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(R/(x_1, ..., x_{i-1})) \setminus V(\mathfrak{q})$ for $i = 1, ..., s$, where $\mathfrak{q} = (x_1, ..., x_s)$ and $V(\mathfrak{q}) = \{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supseteq \mathfrak{q}\}$.

If we denote by $G_J(M) = \bigoplus_{n \geq 0} J^n M / J^{n+1} M$ the associated graded module of M with respect to J, it is well known that $G_J(M)$ is a graded $G_J(R)$-module and set

$$a_i(G_J(M)) = \sup\{n \mid H^i_{G_J(R)^+}(G_J(M))_n \neq 0\}$$

then the Castelnuovo-Mumford regularity of $G_J(M)$ is defined by

$$\operatorname{reg}(G_J(M)) = \max\{a_i(G_J(M)) + i \mid i \geq 0\}.$$

3 Main result

Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$ in R. Suppose that J is a parameter ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. In this section, we will prove that Hilbert coefficients of R/I with respect to J does not change under J-adic perturbations of I. First, we need some useful lemmas for the proof of the main theorem.

The following lemma is implied directly from properties of homological degree.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a noetherian local ring, J an $\mathfrak{m}$-primary ideal of R and M a finitely generated R-module. Let L be a submodule of M with finite length then*

$$\Delta(J, M) = \Delta(J, M/L) + \lambda(L).$$

Proof. Since $\operatorname{hdeg}(J, M)$ is an extended degree of M with respect to J,

$$\operatorname{hdeg}(J, M) = \operatorname{hdeg}(J, M/L) + \lambda(L).$$

Therefore

$$\begin{aligned}\Delta(J, M) &= \operatorname{hdeg}(J, M) - e(J, M) \\ &= \operatorname{hdeg}(J, M/L) + \lambda(L) - e(J, M) \\ &= \Delta(J, M/L) + \lambda(L).\end{aligned}$$

□

The next lemma gives the vanishing of the Castelnuovo-Mumford regularity if J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for all $i = 1, ..., r$.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a noetherian local ring, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$ in R. Suppose that J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Then*

$$\operatorname{reg}(G_J(R/(f_1, ..., f_i)) = 0 \ \text{for} \ \ i = 1, ..., r.$$

Proof. Since J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$, by [6, Corollary 5.2], we have

$$\operatorname{reg}(G_J(R/(f_1, ..., f_i)) = 0 \ \text{for} \ \ i = 1, ..., r.$$

□

Combining Lemma 3.1, Lemma 3.2 and [3, Theorem 3.5 (i)], we have the following corollary.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a noetherian local ring, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$. Suppose that J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Let $\Delta_i = \Delta(J, R/(f_1, ..., f_{i-1})), i = 1, ..., r$ and $\Delta = \max\{\Delta_i \mid i = 1, ..., r\}$. Set*

$$N = (2^r - 1)\Delta + 1.$$

Then for every $\varepsilon_1, ..., \varepsilon_r \in J^N$

i) *$f_1+\varepsilon_1, ..., f_r+\varepsilon_r$ is a filter regular sequence;*

ii) *J is an ideal generated by a d-sequence of R/I', where $I' = (f_1+\varepsilon_1, ..., f_r+\varepsilon_r)$.*

Proof. i) Since $f_1, ..., f_r$ is a filter regular sequence,

$$\lambda((f_1, ..., f_{i-1}) : f_i/(f_1, ..., f_{i-1})) \leq \lambda(H^0_{\mathfrak{m}}(R/(f_1, ..., f_{i-1}))$$

for $i = 1, ..., r$. By the proof [3, Theorem 3.5],

$$a_J((f_1, ..., f_{i-1}) : f_i/(f_1, ..., f_{i-1})) \leq \lambda((f_1, ..., f_{i-1}) : f_i/(f_1, ..., f_{i-1})).$$

But $H^0_{\mathfrak{m}}(R/(f_1, ..., f_{i-1}))$ is a submodule of finite length of $R/(f_1, ..., f_{i-1})$. By Lemma 3.1,

$$\lambda(H^0_{\mathfrak{m}}(R/(f_1, ..., f_{i-1})) \leq \Delta(J, R/(f_1, ..., f_{i-1}))$$

for $i = 1, ..., r$. Hence

$$\sum_{i=1}^{r} 2^{i-1} a_J((f_1, ..., f_{i-1}) : f_i/(f_1, ..., f_{i-1})) \leq (1+2+\cdots+2^{r-1})\Delta = (2^r-1)\Delta.$$

By Lemma 3.2, $\operatorname{reg}(G_J(R/(f_1, ..., f_i))) = 0$ for $i = 1, ..., r$. Therefore

$$\max\{\sum_{i=1}^{r} 2^{i-1} a_J((f_1, ..., f_{i-1}) : f_i/(f_1, ..., f_{i-1})), \operatorname{ar}_J(R/(f_1), ..., \operatorname{ar}_J(R/(f_1, ..., f_r))\} + 1 \leq (2^r-1)\Delta + 1 = N.$$

By [3, Theorem 3.5, (i)], $f_1+\varepsilon_1, ..., f_r+\varepsilon_r$ is a filter regular sequence for all $\varepsilon_1, ..., \varepsilon_r \in J^N$.

ii) By [3, Corollary 3.12 (ii)], $\operatorname{reg}(G_J(R/I')) = \operatorname{reg}(G_J(R/I))$, but $\operatorname{reg}(G_J(R/I)) = 0$, it follows that $\operatorname{reg}(G_J(R/I')) = 0$. By [6, Corollary 5.2], J is an ideal generated by a d-sequence of R/I'. □

The following proposition gives the isomorphism of $I = (f_1, ..., f_r)$ and its perturbation.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$. Suppose that J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Let $\Delta_i = \Delta(J, R/(f_1, ..., f_{i-1})), i = 1, ..., r$ and $\Delta = \max\{\Delta_i \mid i = 1, ..., r\}$. Set*

$$N = (2^r-1)\Delta + 1.$$

Then for every $\varepsilon_1, \ldots, \varepsilon_r \in J^N$,

$$(f_1, ..., f_i) \cong (f_1+\varepsilon_1, ..., f_i+\varepsilon_i)$$

for $i = 1, ..., r$.

Proof. We will prove by induction on r.

In the case $r = 1$. Set $f = f_1$ and $\varepsilon = \varepsilon_1$. We will prove

$$(0 : (f+\varepsilon)) = (0 : f).$$

Let $z \in (0 : f) \subseteq H^0_{\mathfrak{m}}(R), zf = 0$, we have $N \geq \lambda(H^0_{\mathfrak{m}}(R))$, since $\varepsilon \in J^N$, it follows that $z\varepsilon = 0$. So $z(f+\varepsilon) = 0$, hence $z \in (0 : (f+\varepsilon))$. Therefore $(0 : f) \subseteq (0 : (f+\varepsilon))$. Inverserly, by Corollary 3.3, $f+\varepsilon$ is a filter regular element of R, thus $(0 : (f+\varepsilon)) \subseteq H^0_{\mathfrak{m}}(R)$ but $\varepsilon \in J^N$ we get $\varepsilon y = 0$ for all $y \in (0 : (f+\varepsilon))$. So $yf = 0$ for all $y \in (0 : (f+\varepsilon))$. Therefore, $(0 : (f+\varepsilon)) \subseteq (0 : f)$. From the exact sequences

$$0 \longrightarrow (0 : f) \longrightarrow R \longrightarrow (f) \longrightarrow 0$$

and

$$0 \longrightarrow (0 : (f+\varepsilon)) \longrightarrow R \longrightarrow (f+\varepsilon) \longrightarrow 0,$$

we get

$$(f) \cong (f+\varepsilon).$$

In the case $r \geq 2$, by induction we may assume that

$$(f_1, ..., f_{i-1}) \cong (f_1+\varepsilon_1, ..., f_{i-1}+\varepsilon_{i-1}) \tag{3.1}$$

for all $\varepsilon_1, ..., \varepsilon_{i-1} \in J^N$. Consider the exact sequences

$$0 \longrightarrow \frac{(f_1, ..., f_{i-1}) : f_i}{(f_1, ..., f_{i-1})} \longrightarrow \frac{R}{(f_1, ..., f_{i-1})} \longrightarrow \frac{(f_1, ..., f_{i-1}, f_i)}{(f_1, ..., f_{i-1})} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{(f_1, ..., f_{i-1}) : (f_i+\varepsilon_i)}{(f_1, ..., f_{i-1})} \longrightarrow \frac{R}{(f_1, ..., f_{i-1})} \longrightarrow$$

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$$\frac{(f_1, ..., f_{i-1}, f_i + \varepsilon_i)}{(f_1, ..., f_{i-1})} \longrightarrow 0$$

for $i = 1, ..., r$. Apply the above proof for $R/(f_1, ..., f_{i-1})$, we have

$$\frac{(f_1, ..., f_{i-1}) : f_i}{(f_1, ..., f_{i-1})} \cong \frac{(f_1, ..., f_{i-1}) : (f_i + \varepsilon_i)}{(f_1, ..., f_{i-1})}$$

for all $\varepsilon_i \in J^N$. It follows that

$$\frac{(f_1, ..., f_{i-1}, f_i)}{(f_1, ..., f_{i-1})} \cong \frac{(f_1, ..., f_{i-1}, f_i + \varepsilon_i)}{(f_1, ..., f_{i-1})}$$

for all $\varepsilon_i \in J^N$. Therefore

$$(f_1, ..., f_{i-1}, f_i) \cong (f_1, ..., f_{i-1}, f_i + \varepsilon_i) \tag{3.2}$$

for all $\varepsilon_i \in J^N$. From (3.1) and (3.2),

$$(f_1, ..., f_{i-1}, f_i) \cong (f_1 + \varepsilon_1, ..., f_{i-1} + \varepsilon_{i-1}, f_i + \varepsilon_i)$$

for all $\varepsilon_1, ..., \varepsilon_i \in J^N$ and $i = 1, ..., r$. □

From Proposition 3.4, we obtain the following corollary.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$. Suppose that J is an ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Set $s = d - r$ and $\Delta_i = \Delta(J, R/(f_1, ..., f_{i-1})), i = 1, ..., r$. Let $\Delta = \max\{\Delta_i \mid i = 1, ..., r\}$ and*

$$N = (2^r - 1)\Delta + 1.$$

Then for every $\varepsilon_1, \ldots, \varepsilon_r \in J^N$,

$$\lambda(H^0_{\mathfrak{m}}(R/(I + J_k))) = \lambda(H^0_{\mathfrak{m}}(R/(I' + J_k)))$$

where $J_k = (x_1, ..., x_k)$ for $0 \leq k \leq s - 1$ and $I' = (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r)$.

Proof. By Proposition 3.4, for all $\varepsilon_1, ..., \varepsilon_r \in J^N$,

$$(f_1, ..., f_r) \cong (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r).$$

It follows that

$$(f_1, ..., f_r) + J_k \cong (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r) + J_k$$

for $0 \leq k \leq s - 1$. Therefore

$$R/(I + J_k) \cong R/(I' + J_k)$$

for $0 \leq k \leq s - 1$. We get

$$\lambda(H^0_{\mathfrak{m}}(R/(I + J_k))) = \lambda(H^0_{\mathfrak{m}}(R/(I' + J_k)))$$

for $0 \leq k \leq s - 1$. □

Now we are in a position to derive the main result.

Theorem 3.6. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$. Suppose that J is a parameter ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Set $s = d - r$ and $\Delta_i = \Delta(J, R/(f_1, ..., f_{i-1})), i = 1, ..., r$. Let $\Delta = \max\{\Delta_i \mid i = 1, ..., r\}$ and*

$$N = (2^r - 1)\Delta + 1.$$

Then for every $\varepsilon_1, \ldots, \varepsilon_r \in J^N$,

$$e_j(J, R/I) = e_j(J, R/I')$$

for $0 \leq j \leq s$ and $I' = (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r)$.

Proof. Assume that $J = (x_1, ..., x_s)$ is a parameter ideal generated by a d-sequence of R/I. By [7, Theorem 4.1], Hilbert coefficients of R/I with respect to J can be written as the following

$$e_0(J, R/I) = \lambda\left(\frac{R}{I + J}\right) - \lambda\left(\frac{(I + J_{s-1}) : x_s}{I + J_{s-1}}\right);$$

$$(-1)^j e_j(J, R/I) = \lambda\left(\frac{(I + J_{s-j}) : x_{s-j+1}}{I + J_{s-j}}\right) - \lambda\left(\frac{(I + J_{s-j-1}) : x_{s-j}}{I + J_{s-j-1}}\right)$$

for $j = 1, ..., s - 1$;

$$(-1)^s e_s(J, R/I) = \lambda\left(\frac{I : x_1}{I}\right).$$

On the other hand,

$$\lambda\left(\frac{(I + J_k) : x_{k+1}}{I + J_k}\right) = \lambda\left(H^0_{\mathfrak{m}}\left(\frac{R}{I + J_k}\right)\right)$$

for all $1 \le k \le s-1$. By Corollary 3.5, we have

$$\lambda\left(H^0_{\mathfrak{m}}\left(\frac{R}{I+J_k}\right)\right) = \lambda\left(H^0_{\mathfrak{m}}\left(\frac{R}{I'+J_k}\right)\right)$$

for all $1 \le k \le s-1$ and $\varepsilon_1, ..., \varepsilon_r \in J^N$. By Corollary 3.3 (ii), J is a parameter ideal generated by a d-sequence of R/I', therefore

$$\lambda\left(\frac{(I'+J_k):x_{k+1}}{I'+J_k}\right) = \lambda\left(H^0_{\mathfrak{m}}\left(\frac{R}{I'+J_k}\right)\right)$$

for all $1 \le k \le s-1$ and $\varepsilon_1, ..., \varepsilon_r \in J^N$. It follows that

$$\lambda\left(\frac{(I+J_k):x_{k+1}}{I+J_k}\right) = \lambda\left(\frac{(I'+J_k):x_{k+1}}{I+J_k}\right)$$

for all $1 \le k \le s-1$ and $\varepsilon_1, ..., \varepsilon_r \in J^N$. On the other hand, by Proposition 3.4,

$$\lambda(R/(I+J)) = \lambda(R/(I'+J))$$

for all $\varepsilon_1, ..., \varepsilon_r \in J^N$. Therefore

$$e_j(J, R/I) = e_j(J, R/I')$$

for all $0 \le j \le s$. □

From Theorem 3.6 and [7, Theorem 4.1], we have the following corollary.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d, J an $\mathfrak{m}$-primary ideal of R and $I = (f_1, ..., f_r)$ an ideal generated by a filter regular sequence $f_1, ..., f_r$. Suppose that J is a parameter ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$. Set $\Delta_i = \Delta(J, R/(f_1, ..., f_{i-1})), i = 1, ..., r$. Let $\Delta = \max\{\Delta_i \mid i = 1, ..., r\}$ and*

$$N = (2^r - 1)\Delta + 1.$$

Then for every $\varepsilon_1, \ldots, \varepsilon_r \in J^N$,

$$\lambda(R/(I+J^{n+1})) = \lambda(R/(I'+J^{n+1}))$$

for all $n \ge 0$ and $I' = (f_1 + \varepsilon_1, ..., f_r + \varepsilon_r)$.

Remark 3.8. In the case J is a parameter ideal generated by a d-sequence of $R/(f_1, ..., f_i)$ for $i = 1, ..., r$, Corollary 3.7 gives a linear bound for the Hilbert perturbation index in terms of the Cohen-Macaulay deviation.

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