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Existence and linear conditioning for solutions of equilibrium problems in metric spaces

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Abstract. In this paper, we provide sufficient conditions for the existence and linear conditioning of equilibrium problems in metric spaces. Our results improve and generalise some well-known results in the literature.

Keywords: existence of solutions, linear conditioning, equilibrium problems, metric spaces

1 Introduction

Let (M, d) be a metric space, X be a nonempty closed subset of M, and $f: X \times X \to \mathbb{R}$ be a bifunction. We consider the following equilibrium problem (in short, EP): Find $x^* \in X$ such that

$$f(x^*, y) \geq 0 \text{ for all } y \in X. \quad (1)$$

We denote the solution set of EP (1) with X^*.

The equilibrium problem was introduced by Blum and Oettli [8]. It is a general mathematical model which contains several important problems as special cases, such as optimisation problems, variational inequality problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems in noncooperative games, and others (see, e.g., [7, 8, 14] and references therein). Since EP has applications in numerous areas of science, such as economics, transportation, networks, image reconstruction, and elasticity, it has been studied extensively. Two basic and important issues for EP are the existence of solutions and solution methods. There have been a large number of papers dealing with the solution existence and algorithms for solving equilibrium problems in the literature (see [5–8, 10–12, 21, 23] and references therein).

Moudafi [17] introduced the notion of θ-conditioning for equilibrium problems. This notion generalises and unifies several known notions and conditions in the literature, concerning optimisation problems, such as the notion of conditioning for functions [15], the notion of sharp minimum [20, 2], the notion of weak sharp minima [9, 13, 16], and conditions for non-expansive mappings [22]. Note that the above-mentioned notions and conditions play a crucial role in the treatment of error bounds, sensitivity analysis, as well as (finite) convergence analysis for a wide range of algorithms for solving optimisation problems, variational inequality problems, and fixed point problems (see, e.g., [4, 16–19] and references therein). It is worth mentioning that Moudafi [17] only gave the definition of θ-conditioning and an application to the finite convergence of the proximal method for solving equilibrium problems. Some characterisations and applications of 1-conditioning to establish finite convergence of some algorithms for solving equilibrium problems in Hilbert spaces were presented in [18].

The main objective of this paper is to study the existence and linear conditioning for solutions of equilibrium problems in metric spaces. We first present an existence result for solutions of equilibrium problems in metric spaces, improving a result by Blum and Oettli [8]. We then use this result to establish some characterisations for linearly conditioned solutions of equilibrium problems. These results improve and extend the analogous results in [18] to the setting of metric spaces.

The remainder of this section is devoted to the presentation of some definitions and basic results which will be used in the following section.

Let K be a subset of M and $x \in M$. The distance from x to K is defined by

$$d(x, K) = \inf_{y \in K} d(x, y).$$

The metric projection from x to K is defined by

$$P_K(x) = \{y \in M : d(x, y) = d(x, M)\}.$$

In general, P_K is a set-valued mapping from M to K. When P_K is single-valued, the set K is called a Chebyshev set. For example, if M is a Hadamard manifold, and d is the Riemannian distance, then every closed, geodesically convex set is a Chebyshev set (see, e.g., [23]).

Definition 1 *Let $\varphi: M \to \mathbb{R}$ be a function and $\hat{x} \in M$. We say that φ is lower semicontinuous at $\hat{x}$ if*

$$f(\hat{x}) \le \liminf_{x \to \hat{x}} f(x).$$

If φ is lower semicontinuous at every $\hat{x} \in M$, we say that φ is lower semicontinuous.

Definition 2. *Let $\varphi: M \to \mathbb{R}$ be a function and $\hat{x} \in M$. We say that φ is 0-lower semicontinuous at $\hat{x}$ if*

$$\liminf_{x \to \hat{x}} f(x) \le 0 \implies f(\hat{x}) \le 0.$$

If φ is 0-lower semicontinuous at every $\hat{x} \in M$, we say that φ is 0-lower semicontinuous.

Remark 1 If φ is lower semicontinuous at a point $x \in M$, then it is 0-lower semicontinuous at x. However, the converse is not true.

Example 1 Let $\varphi: \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} 0 & if\ x \le 0, \\ x^2 - 1 & if\ x > 0. \end{cases}$$

One can check that φ is 0-lower semicontinuous at 0, but it is not lower semicontinuous at 0.

Definition 3 *A bifunction $f: X \times X \to \mathbb{R}$ is said to be*

(a) monotone on X if, for any $x, y \in X$,

$$f(x, y) + f(y, x) \le 0;$$

(b) pseudomonotone on X if, for any $x, y \in X$,

$$f(x, y) \ge 0 \implies f(y, x) \le 0.$$

From the definition, we see that if f is monotone, then it is pseudomonotone. However, the converse is not true even in Hilbert spaces (see, e.g., [12]).

The following result was stated in [18], where the authors studied equilibrium problems in Hilbert spaces. We present the proof for the reader's convenience.

Lemma 1 If $f: X \times X \to \mathbb{R}$ is pseudomonotone, then $f(x^*, y^*) = 0$ for all $x^*, y^* \in X^*$.

Proof. Since $y^* \in X^*$, one has $f(y^*, x^*) \ge 0$. Then, by the pseudomonotonicity of f,

$$f(x^*, y^*) \le 0.$$

Moreover, since $x^* \in X^*$, one also has

$$f(x^*, y^*) \ge 0.$$

It follows from the last two inequalities that $f(x^*, y^*) = 0$. This ends the proof. ■

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2 Main results

2.1 Existence of solutions

In this subsection, we present an existence theorem for solutions of equilibrium problems. The result slightly refines a result by Blum and Oettli [8, Theorem 3].

Theorem 1 *Let X be a nonempty closed subset of M and $f: X \times X \to \mathbb{R}$ satisfy the following conditions:*

(i) *f is 0-lower semicontinuous in the second argument;*

(ii) $f(x,y) \le f(x,z) + f(z,y)\ \forall x,y,z \in X$;

(iii) *there exists $x_0 \in X$ such that*

$\inf_{y\in X} f(x_0,y) > -\infty$;

(iv) there exists $\gamma > 0$ such that for every $x \in X$ with $\inf_{y\in X} f(x,y) < 0$, there is some $y \in X, y \ne x$ satisfying

$f(x,y) + \gamma d(x,y) \le 0$.

Then, there exists $x^* \in X$ such that $f(x^*,y) \ge 0$ for all $y \in X$, i.e., the solution set X^* of EP(1) is nonempty.

Proof. The proof is a slight modification of the proof of Theorem 3 in [8]. We present it here for the reader's convenience.

We will construct, by induction, a sequence $\{x_n\} \subset X$ starting from x_0 such that for all n,

$$x_{n+1} \in S_n, \qquad f(x_n, x_{n+1}) \le \alpha_n + \frac{1}{2^n}, \tag{2}$$

and

$$\emptyset \ne S_{n+1} \subset S_n \tag{3}$$

where

$$S_n = \{y \in X : f(x_n,y) + \gamma d(x_n,y) \le 0\}$$

and

$$\alpha_n = \inf_{y\in S_n} f(x_n,y) > -\infty. \tag{4}$$

Since $f(x,x) = 0$ for all $x \in X$, we have $x_0 \in S_0$, i.e., $S_0 \ne \emptyset$. Moreover, by (iii),

$$\alpha_0 = \inf_{y\in S_0} f(x_0,y) \ge \inf_{y\in X} f(x_0,y) > -\infty.$$

Then, there exists $x_1 \in S_0$ such that

$f(x_0,x_1) \le \alpha_0 + 1$.

By (ii) and the fact that $x_1 \in S_0$, we have, for each $y \in S_1$,

$$f(x_0,y) + \gamma d(x_0,y) \le f(x_0,x_1) + f(x_1,y) + \gamma d(x_0,x_1) + \gamma d(x_1,y) \le 0.$$

This implies that $y \in S_0$. So, $S_1 \subset S_0$. Of course, $S_1 \ne \emptyset$ since $x_1 \in S_1$. Hence, (2), (3) and (4) hold for $n = 0$.

Assume now that, for some $k \ge 0$, we have constructed $x_0, x_1, \cdots, x_k, x_{k+1}$ such that (2), (3), and (4) hold for $n = 0,1,\cdots,k$. By (ii) and the inductive hypothesis, we have

$$\begin{aligned}\alpha_{n+1} &= \inf_{y\in S_{n+1}} f(x_{n+1},y)\\ &\ge \inf_{y\in S_{n+1}} [f(x_n,y) - f(x_n,x_{n+1})]\\ &\ge \inf_{y\in S_n} [f(x_n,y) - f(x_n,x_{n+1})]\\ &= \alpha_n - f(x_n,x_{n+1}) \ge -\frac{1}{2^n}.\end{aligned}$$

This implies that $\alpha_{n+1} > -\infty$. This leads to the existence of an element $x_{n+2} \in S_{n+1}$ such that

$$f(x_{n+1},x_{n+2}) \le \alpha_{n+1} + \frac{1}{2^{n+1}}.$$

It is easy to see that $S_{n+2} \subset S_{n+1}$ and $x_{n+2} \in S_{n+2}$. Therefore, by induction, the construction of the sequence $\{x_n\}$ satisfying (2), (3) and (4) is complete.

We see that for each n, the set S_n is closed. Indeed, assume that $\{y_k\} \subset S_n$ converges to some $y \in X$. Since $y_k \in S_n$,

$f(x_n,y_k) + \gamma d(x_n,y_k) \le 0$.

This implies that

$$\liminf_{k\to\infty} [f(x_n,y_k) + \gamma d(x_n,y_k)] \le 0.$$

By (i), we get $f(x_n,y) + \gamma d(x_n,y) \le 0$. This means that $y \in S_n$. So, S_n is closed.

Now, for each $n \ge 1$, if $y \in S_n$, then

$$d(x_n, y) \le -\frac{1}{\gamma} f(x_n, y) \le -\frac{1}{\gamma} \alpha_n \le \frac{1}{\gamma} . \frac{1}{2^{n-1}}.$$

This implies that the diameter of the sets S_n tends to zero. Moreover, $x_k \in S_k \subset S_n$ for all $k \ge n$. Thus, $d(x_n, x_k) \le \frac{1}{\gamma} . \frac{1}{2^{n-1}}$.

This means that $\{x_n\}$ is a Cauchy sequence in X. Since X is closed, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

It is evident that

$$\{x^*\} = \bigcap_{n=0}^{\infty} S_n. \quad (5)$$

We claim that $f(x^*, y) \ge 0$ for all $y \in X$. If this is not true, then, by (iv), there exists $y^* \in X$ with $y^* \neq x^*$ such that

$$f(x^*, y^*) + \gamma d(x^*, y^*) \le 0. \quad (6)$$

By (5), $f(x_n, x^*) + \gamma d(x_n, x^*) \le 0$ for all n. Using (ii), (6) one has $f(x_n, y^*) + \gamma d(x_n, y^*) \le 0$. This implies that $y^* \in \bigcap_{n=0}^{\infty} S_n$. This contradicts (5). The proof is complete. ∎

Remark 2 Under the assumptions of Theorem 1, $x^* \in X^*$ if and only if

$$f(x^*, y) + \gamma d(x^*, y) > 0 \ \forall y \in X, y \neq x. \quad (7)$$

Indeed, if $x^* \in X^*$, then $f(x^*, y) > 0$ for all $y \in X$. This implies that (7) holds. Suppose now that (7) holds. We show that $x^* \in X^*$. If not, there is some $u \in X$ such that $f(x^*, u) < 0$. Then, by (iv), there exists some $y \in X, y \neq x$ such that

$$f(x, y) + \gamma d(x, y) \le 0.$$

This contradicts (7). Thus, $x^* \in X^*$.

Remark 3 Theorem 1 improves [8, Theorem 3] in the sense that we replace the lower semicontinuity of f by the 0-lower semicontinuity.

2.2 Linear conditioning of solution sets

This subsection is devoted to the study of some sufficient conditions for the linear conditioning property of solution sets of equilibrium problems in metric spaces. Throughout this subsection, we always assume that the solution set X^* is a Chebyshev set. We refer the reader to [23] for some sufficient conditions under which X^* is closed, geodesically convex (i.e., X^* is a Chebyshev set) in case M is a Hadamard manifold.

Definition 4 [17] The *equilibrium bifunction f is said to be θ-conditioned with modulus γ if there exist two positive constants θ and γ such that*

$$-f\big(x, P_{X^*}(x)\big) \ge \gamma[d(x, X^*)]^{\theta} \ \ \forall x \in X. \quad (8)$$

We say that f is linearly conditioned with modulus γ if it is 1-conditioned with modulus γ.

Remark 4 We also say that the solution set X^* is θ-conditioned with modulus γ if (8) holds. When $\theta = 1$, the solution set X^* is said to be linearly conditioned with modulus γ.

We now present some sufficient conditions for the linear conditioning of the solution set X^* of EP (1). The following theorem extends [18, Proposition 2] to the setting of metric spaces with a more detailed proof.

Theorem 2 *Suppose the bifunction $f: X \times X \to \mathbb{R}$ is pseudomonotone on X and satisfies the assumptions (i)–(iv) in Theorem 1. Then, the solution set X^* of EP(1) is linearly conditioned with modulus γ.*

Proof. By Theorem 1, the solution set X^* of EP (1) is nonempty. For each $x \in X$, set

$$S(x) = \{y \in X: f(x, y) + \gamma d(x, y) \le 0\}.$$

We have that $S(x)$ is nonempty since $x \in S(x)$. Moreover, since f is 0-lower semicontinuous on X, $S(x)$ is a closed set for each $x \in X$. Using (ii) and (iii), we have

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$$\inf_{y\in S(x)} f(x,y) \geq \inf_{y\in S(x)}[f(x_0,y)-f(x_0,x)] \geq \inf_{y\in X}[f(x_0,y)-f(x_0,x)] > -\infty.$$

Now, by applying Theorem 1 with X being replaced by $S(x)$, there exists $u^* \in S(x)$ such that $f(u^*,y) \geq 0$ for all $y \in S(x)$. By Remark 2, we have

$$f(u^*,y)+\gamma d(u^*,y) > 0, \forall y \in S(x), y \neq u^*. \quad (9)$$

We need to show that

$$f(u^*,y)+\gamma d(u^*,y) > 0, \forall y \in X, y \neq u^* \quad (10)$$

If (10) does not hold, then there exists $u \in X, u \neq u^*$ such that

$$f(u^*,u)+\gamma d(u^*,u) \leq 0. \quad (11)$$

This, together with the fact $u^* \in S(x)$, implies that

$$f(x,u)+\gamma d(x,u) \leq f(x,u^*)+\gamma d(x,u^*) + f(u^*,u)+\gamma d(u^*,u) \leq 0.$$

Hence, $u \in S(x)$. This is a contradiction since (9) with $y = u$ and (11) do not hold simultaneously. Thus, (10) holds, and by Remark 2, $u^* \in X^*$. This implies that $S(x) \cap X^* \neq \emptyset$.

Now, for each $x \in X\backslash X^*$, let $z \in S(x) \cap X^*$ (z depends on x). We have

$$f(x,z)+\gamma d(x,X^*) \leq f(x,z)+\gamma d(x,z) \leq 0.$$

This implies that

$$\gamma d(x,X^*) \leq -f(x,z).$$

This, together with (ii) and Lemma 1, leads to

$$\gamma d(x,X^*) \leq -f(x,z) \leq -f\big(x,P_{X^*}(x)\big)+f\big(z,P_{X^*}(x)\big) = -f\big(x,P_{X^*}(x)\big).$$

This implies that X^* is linearly conditioned with modulus γ. ■

The following theorem improves and generalises [18, Proposition 3] by extending the result from Hilbert spaces to metric spaces, and by replacing the assumption of lower semicontinuity of φ with the weaker condition of 0-lower semicontinuity.

Theorem 3 *Let $f: X\times X \to \mathbb{R}$ be a monotone bifunction. Assume that there exists a function $\varphi: X \to \mathbb{R}$ and a constant $\gamma > 0$ such that*

(a) φ is 0-lower semicontinuous and is bounded below;

(b) $f(x,y) \geq \varphi(y)-\varphi(x)$ for all $x, y \in X$;

(c) for each $\hat{x} \in X$ with $\inf_{x\in X}\varphi(x) < \varphi(\hat{x})$, there exists $z \in X$ such that $z \neq \hat{x}$ and

$$f(\hat{x},z)+\gamma d(\hat{x},z) \leq 0.$$

Then, f is linearly conditioned with modulus γ.

Proof. Denote by E the set of minimisers of φ. Then, by condition (b), $E \subset X^*$. For reach $x \in X$, set

$$S(x) = \{y \in X: \varphi(y)-\varphi(x)+\gamma d(y,x) \leq 0\}.$$

Then, $S(x)$ is nonempty since $x \in S(x)$. Moreover, since φ is 0-lower semicontinuous, the set $S(x)$ is closed. Thus, $(S(x),d)$ is a complete metric space. Now, by applying Theorem 1 and using Remark 2 for $f(x,y) = \varphi(y)-\varphi(x)$, there exists $x^* \in S(x)$ such that

$$\varphi(y)-\varphi(x^*)+\gamma d(x^*,y) > 0, \forall y \in S(x), y \neq x.$$

As in the proof of Theorem 2, we can show that

$$\varphi(y)-\varphi(x^*)+\gamma d(x^*,y) > 0, \forall y \in X, y \neq x. \quad (12)$$

We claim that $x^* \in E$. Indeed, if $x^* \notin E$, then by (c), there exists $z \in X$ with $z \neq x^*$ such that

$$f(x^*,z)+\gamma d(x^*,z) \leq 0.$$

This, together with (b), implies that

$$\varphi(z)-\varphi(x^*)+\gamma d(x^*,z) \leq 0.$$

This contradicts (12). So, $x^* \in E \subset X$ and hence $x^* \in S(x) \cap X^*$. This means that for each $x \in X$, there exists $x^* \in X^*$ (depending on x) such that

$$\gamma d(x,x^*) \leq \varphi(x)-\varphi(x^*).$$

Using (ii), Lemma 1 and the monotonicity of f, one has for each $x \in X$ that

$$\begin{aligned}\gamma d(x, X^*) \le \gamma d(x, x^*) &\le \varphi(x) - \varphi(x^*)\\ &= \varphi(x) - \varphi\big(P_{X^*}(x)\big)\\ &\quad + \varphi\big(P_{X^*}(x)\big) \quad - \varphi(x^*)\\ &\le f(P_{X^*}(x), x) + f\big(x^*, P_{X^*}(x)\big)\\ &\le -f\big(x, P_{X^*}(x)\big).\end{aligned}$$

Here, according to Lemma 1, $f\big(x^*, P_{X^*}(x)\big) = 0$ as $x^*, P_{X^*}(x) \in X^*$. The latter inequality means that X^* is linearly conditioned with modulus γ. ∎

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References

1. Al-Homidan S, Ansari QH, Kassay, G. Takahashi's minimization theorem and some related results in quasi-metric spaces. Journal of Fixed Point Theory and Applications. 2019;21(1):38.
2. Al-Homidan S, Ansari QH, Nguyen LV. Finite convergence analysis and weak sharp solutions for variational inequalities. Optimization Letters. 2017;11(8):1647-62.
3. Alshahrani M, Al-Homidan S, Ansari QH. Weak sharp solutions for equilibrium problems in metric spaces. J Nonlinear Convex Anal. 2015;16(7);1185-1193.
4. Antipin AS. On finite convergence of processes to a sharp minimum and to smooth minimum with a sharp derivative. Differ Equ. 1994;30:1703-1713.
5. Babu F, Ali A, Alkhaldi AH. An extragradient method for non-monotone equilibrium problems on Hadamard manifolds with applications. Applied Numerical Mathematics. 2022;180:85-103.
6. Bento GdC, Cruz Neto JX, Melo ÍDL. Combinatorial convexity in Hadamard manifolds: existence for equilibrium problems. Journal of Optimization Theory and Applications. 2022;195(3):1087-105.
7. Bigi G, Castellani M, Pappalardo M, Passacantando M. Existence and solution methods for equilibria. European Journal of Operational Research. 2013;227(1):1-11.
8. Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems. Math Student. 1994;62(1):127-169.
9. Burke JV, Ferris MC. Weak sharp minima in mathematical programming. SIAM Journal on Control and Optimization. 1993;31(5):1340-59.
10. Colao V, López G, Marino G, Martín-Márquez V. Equilibrium problems in Hadamard manifolds. Journal of Mathematical Analysis and Applications. 2012;388(1):61-77.
11. Combettes PL, Hirstoaga A. Equilibrium programming in Hilbert spaces. J Nonlinear Convex Anal. 2005;6:117-136.
12. Cruz Neto J, Santos P, Soares P. An extragradient method for equilibrium problems on Hadamard manifolds. Optim. Lett. 2016; 10(6):1327–1336.
13. Ferris MC. Weak Sharp Minima and Penalty Functions in Mathematical Programming [dissertation]. Cambridge: University of Cambridge; 1988.
14. Konnov I. Equilibrium Models and Variational Inequalities. Amsterdam: Elsevier; 2007.
15. Lemaire B. Well-Posedness, conditioning and regularization of minimization, inclusion and fixed-point problems. Pliska Stud Math Bulgar. 1998;12: 71-84.
16. Marcotte P, Zhu DL. Weak sharp solutions of variational inequalities. SIAM Journal on Optimization. 1998;9(1):179-89.
17. Moudafi A. On finite and strong convergence of a proximal method for equilibrium problems. Numerical Functional Analysis and Optimization. 2007;28(11-12):1347-54.
18. Nguyen LV, Ansari QH, Qin X. Linear conditioning, weak sharpness and finite convergence for equilibrium problems. Journal of Global Optimization. 2020;77(2):405-24.
19. Nguyen LV, Thu NT, An NT. Characterizations of weak sharp solutions for variational inequalities on Hadamard manifolds and applications. Journal of Industrial and Management Optimization. 2024;20(2):729-51.
20. Polyak BT. Sharp Minima, Institute of Control Sciences Lecture Notes, Moscow, USSR (1979). Presented at the IIASA Workshop on Generalized

pISSN 1859-1388
eISSN 2615-9678

Lagrangians and Their Applications, IIASA, Laxenburg, Austria; 1979.

21. Quoc TD, Muu LD. Iterative methods for solving monotone equilibrium problems via dual gap functions. Computational Optimization and Applications. 2012;51(2):709-28.
22. Senter HF, Dotson WG. Approximating fixed points of nonexpansive mappings. Proc Am Math Soc. 1974; 44:375-380.
23. Wang X, López G, Li C, Yao J-C. Equilibrium problems on Riemannian manifolds with applications. Journal of Mathematical Analysis and Applications. 2019;473(2):866-91.